Inventiones math. 53, 1–44 (1979)

*Inventiones mathematicae*
© by Springer-Verlag 1979

# Intersection Numbers of Sections of Elliptic Surfaces

David A. Cox and Steven Zucker*

Department of Mathematics, Rutgers University, New Brunswick, N.J. 08903, USA

The theory of elliptic surfaces over $\mathbf{C}$ draws on ideas and techniques from arithmetic, geometry and analysis. Let $\bar{f}: \bar{X} \to \bar{S}$ be a minimal elliptic fibration with non-constant $j$-invariant, which possesses a section $\sigma_0$. Then the group $\mathfrak{S}$ of sections of $\bar{f}$ can be naturally identified with the group of rational points of the generic fiber; i.e., $\mathfrak{S}$ consists of rational solutions of a cubic equation over a function field. On the analytic end, one can associate to these sections certain generalized automorphic forms which have a natural pairing (the Eichler bilinear form) defined on them. In this paper, we will examine the connection of these topics with the geometry of $\bar{X}$ via the intersection product and Hodge decomposition on its cohomology.

This paper had its origin in a problem posed by W. Hoyt: if the rank $r$ of the group $\mathfrak{S}$ of sections is known (e.g., when $p_g = 0$) and one has $r$ sections $\sigma_1, \ldots, \sigma_r$, do they form a basis of $\mathfrak{S}$ modulo torsion? The earlier attempt by Hoyt and Schwartz to answer this question involved a direct use of the Eichler pairing. Its values lie in $(1/N)\mathbf{Z}$ for some $N \in \mathbf{Z}$, and a bound for the best value of $N$ can be computed from the monodromy. Thus, by looking at the discriminant of this form on the given sections, one obtains a sufficient (though not always necessary) condition for them to be generators modulo torsion. The hope had been that one would be able to compute this discriminant in examples. However, the calculations are very messy and have been done only in the simplest of cases. Another difficulty in this program is that sometimes one must base-change before the Eichler pairing can be defined.

We give an alternate method (using intersection numbers on $\bar{X}$) for doing such computations (§ 1), and we will use it to compute a large number of examples (§ 2). In § 3 we will show how our geometric methods relate to the automorphic forms and Eichler pairing, and we also determine the role they play in the Hodge theory of $\bar{X}$.

Now we discuss the contents of the paper in greater detail. Given $\bar{f}: \bar{X} \to \bar{S}$ as described above, let $f: X \to S$ be the smooth part of $\bar{f}$ (throw away the singular

* Supported in part by NSF Grant MCS 76-06364

0020-9910/79/0053/0001/$08.80

fibers), and let $j\colon S\to\bar{S}$ be the inclusion. In § 1 we define a homomorphism:

$$\delta\colon\ \mathfrak{S}\to H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})\cong H^1(\bar{S}, j_*R^1f_*\mathbf{Q})$$

as follows. Given $\sigma\in\mathfrak{S}$, we want to alter its cycle class $[\sigma]\in H^2(\bar{X},\mathbf{Z})$ so that it lies in the first Leray filtration level $L^1$ (which consists of classes that restrict to zero on all of the fibers); in general, this can be done only if we pass to rational cohomology. The class $[\sigma-\sigma_0]$ gives us zero on all the good fibers, but if $\sigma$ and $\sigma_0$ pass through different components of a singular fiber, some more corrections are needed. This is done by adding a *rational* linear combination $D$ of components of singular fibers (and $D$ is determined at each fiber solely by the component that $\sigma$ hits – see (1.14)). Having now $\delta(\sigma)=[\sigma-\sigma_0+D]\in L^1$, we define a bilinear pairing $\langle\ ,\ \rangle$ on $\mathfrak{S}$ by setting:

$$\langle\sigma,\sigma'\rangle=-(\delta(\sigma)\cup\delta(\sigma'))$$

for $\sigma,\sigma'$ in $\mathfrak{S}$ (and the cup product is taken equivalently in $H^2(\bar{X},\mathbf{Q})$ or in $H^1(\bar{S},R^1\bar{f}_*\mathbf{Q})$). The pairing differs surprisingly little from the intersection number:

(0.1) $[\sigma-\sigma_0]\cdot[\sigma'-\sigma_0]$

and the difference is again expressible in terms of the components of the bad fibers hit by the sections. The correction factors are listed in (1.19).

We have $\langle\sigma,\sigma\rangle\geqq 0$, and $\langle\sigma,\sigma\rangle=0$ if and only if $\sigma$ is torsion. The discriminant of $\langle\ ,\ \rangle$ on $\mathfrak{S}$ is easily computable in terms of the torsion of $\mathfrak{S}$, the singular fibers of $\bar{f}$, and the discriminant of intersection product on the Néron-Severi group $NS(\bar{X})$. Normally one does not know much about $NS(\bar{X})$, but when $p_g=0$, $NS(\bar{X})=H^2(\bar{X},\mathbf{Z})$ and the discriminant of intersection product is 1 by Poincaré duality. Then sections $\sigma_1\ldots\sigma_r$ of $\bar{f}$, where $r=\operatorname{rank}\mathfrak{S}$, are a basis of $\mathfrak{S}$ (modulo torsion) if and only if:

(0.2) $\det\langle\sigma_i,\sigma_j\rangle=(\#\mathfrak{S}_{\mathrm{tor}})^2/\prod m_s$

where $m_s$ is the number of components of multiplicity one in the fiber $X_s=\bar{f}^{-1}(s)$. This formula is the basis for all of the examples in § 2. Then we examine the discriminant of cup product on $H^1(\bar{X},R^1\bar{f}_*\mathbf{Z})$ and discuss (without proof) the relation between $\mathfrak{S}_{\mathrm{tor}}$ and $H^2(\bar{S},R^1\bar{f}_*\mathbf{Z})$. The section ends with a treatment of the relation between $\langle\ ,\ \rangle$ and Tate heights.

In the Appendix to § 1 we consider *arbitrary* elliptic fibrations (possibly non-algebraic) $\bar{f}\colon\bar{X}\to\bar{S}$, and we determine sufficient (and often necessary) conditions for the maps $\bar{f}_*\colon\pi_1(\bar{X})\to\pi_1(\bar{S})$ and $\bar{f}_*\colon H_1(\bar{X},\mathbf{Z})\to H_1(\bar{S},\mathbf{Z})$ to be isomorphisms. In this paper we use these results to conclude that various cohomology groups are torsion-free.

§ 2 has a strong arithmetic flavor to it, since $\mathfrak{S}$ is isomorphic to the group of rational solutions of any Weierstrass equation:

(0.3) $y^2=4x^3-g_2x-g_3,\qquad g_2,g_3\in K(\bar{S})$

defining the generic fiber of $\bar{f}$. Typical of the examples we give is: the $\mathbf{C}(t)$-

rational solutions of $y^2=4x^3-3t^3x+t^4$ form an infinite cyclic group generated by $(0, t^2)$. All of the examples are surfaces with $p_g=0$ (so that (0.2) is true and $r=\operatorname{rank}\mathfrak{S}$ is known). Each example starts with explicit solutions $\sigma_1, \ldots, \sigma_r$ of an Eq. (0.3), which are shown to be a basis of $\mathfrak{S}$ (modulo torsion) by using (0.2). Most of the machinery for computing $\langle\sigma, \sigma'\rangle$ is developed in § 1 (see (1.18) and (1.19)), with two exceptions. First, we need to determine which component of a bad fiber is hit by a section. This requires looking at the bad fibers case by case (using the Kodaira classification) and it occupies a large part of § 2. Second, computing the intersection product (0.1) is complicated by the fact that all we have to work with is the Weierstrass equation, which does not define $\bar{X}$ at the singular fibers. So we must show how to avoid this difficulty. We also show how to compute $\mathfrak{S}_{\mathrm{tor}}$, which is necessary for (0.2). Two of the examples are worked out in detail.

In § 3, we treat the relation between automorphic forms and the cohomology group $H^1(\bar{S}, j_* V)$, where $V=R^1 f_* \mathbf{C}$. We begin by observing that this group is naturally isomorphic to the first parabolic cohomology group associated to the monodromy representation of $\pi_1(S)$ on the first cohomology group of the fiber. A section $\sigma\in\mathfrak{S}$ gives rise to elements of $H^1(\bar{S}, j_* V)$ in three seemingly different ways:

1. The class $\delta(\sigma)$ from § 1.

2. Lifting $\sigma$ to the upper half-plane $\mathfrak{h}$ (the universal cover of $S$), $\sigma$ becomes expressible as an analytic function $F\colon \mathfrak{h}\to\mathbf{C}$, which possesses a period cocycle.

3. Let $\tau\colon \mathfrak{h}\to\mathfrak{h}$ denote the period function for $X/S$, lifted to $\mathfrak{h}$. Then one obtains a generalized automorphic form of weight 3 (3/2 to some) from $F$ above by differentiating twice with respect to $\tau$. Such automorphic forms feed naturally into the parabolic cohomology.

It is not hard to see that in fact all three are equal ((3.9) and (3.10)). Using this, we prove (3.12) that the generalized Eichler pairing for the automorphic forms coming from sections $\sigma$ and $\sigma'$ is none other than our pairing $\langle\sigma, \sigma'\rangle$. Consequently, we are working with the same pairing as Hoyt and Schwartz, with a more efficient way of evaluating it.

We conclude (§ 3C) with results on the Hodge decomposition of $H^1(\bar{S}, j_* V)$. By [25], there is a filtered complex comprised of locally-free $\mathcal{O}_{\bar{S}}$-modules (extending $\Omega_S^{\cdot}(V)$) whose hyper-cohomology gives the cohomology of $j_* V$, such that the induced filtration on cohomology gives a Hodge structure of weight two:

$$H^1(\bar{S}, j_* V)=H^{2,0}\oplus H^{1,1}\oplus H^{0,2};$$

moreover, it coincides with the one induced by $H^2(\bar{X}, \mathbf{C})$ through the Leray spectral sequence. Using this, we show ((3.20) and (3.24)) that filtration levels $F^2=H^{2,0}$ and $F^1=H^{2,0}\oplus H^{1,1}$ are naturally isomorphic to certain spaces of automorphic forms. The description of $F^1$ involves the ramification divisor of $\tau$ on $\bar{S}$. In this, we must make a reasonable extension to $\Sigma=\bar{S}-S$ of the notion of ramification (3.17). Then $\dim H^{1,1}$ is equal to the total order of ramification of $\tau$, which yields a clearer proof (3.21) of a result in Shioda's paper [23]. Similar in spirit, we reprove in (3.22) another proposition from [23], giving an upper bound on the rank of $\mathfrak{S}$.

We are grateful to W. Hoyt for getting us interested in the problem described above. He conjectured some of the results proved in this paper. The idea of using a correction term consisting of a rational linear combination of components of bad fibers was suggested by P. Deligne. We would also like to thank G. Winters and R. Miranda for several useful conversations.

## § 1. The Pairing on Sections

An elliptic fibration over $\mathbf{C}$ is a map $\bar{f}\colon \bar{X}\to\bar{S}$ where $\bar{S}$ is a smooth compact curve over $\mathbf{C}$, $\bar{X}$ is a smooth compact surface over $\mathbf{C}$, and the generic fiber of $\bar{f}$ is an elliptic curve. We will assume that $\bar{f}$ has a section $\sigma_0$ and that there are no exceptional curves of the first kind in the fibers of $\bar{f}$; then $\bar{f}\colon \bar{X}\to\bar{S}$ is the Néron model of its generic fiber (and $\bar{f}$ is what Kodaira calls the "basic member"); see [14] and [8, § 8]. We will also assume that the $j$-invariant of $\bar{f}$ is non-constant.

The fiber $\bar{f}^{-1}(s)$, $s\in\bar{S}$, will be denoted $X_s$, and we let $S=\{s\in\bar{S},\ X_s$ is a non-singular fiber$\}$. The set $\Sigma=\bar{S}-S$ is finite, and the $X_s$, $s\in\Sigma$ are the "bad fibers" of $\bar{f}$. We will use Kodaira's classification of the bad fibers into types $\mathrm{I}_b$ $(b>0)$, $\mathrm{I}_b^*$ $(b\geqq 0)$, II, II*, III, III*, IV, and IV* (see [8, § 6]).

Let $X$ be the inverse image of $S$ in $\bar{X}$, so that we have a commutative diagram:

$$(1.1)\qquad \begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ S & \xhookrightarrow{j} & \bar{S} \end{array}$$

where $f$ is proper and smooth. For an abelian group $M$, the locally constant sheaf $V_M=R^1 f_* M$ on $S$ corresponds to the monodromy representation of $\pi_1(S)$ on $H^1(X_t, M)$ (where $t\in S$). $M$ will be either $\mathbf{Z}, \mathbf{Q}$ or $\mathbf{C}$, and $V_{\mathbf{C}}$ will be denoted simply $V$. By abuse of notation we will sometimes write $V_M=H^1(X_t, M)$.

Finally we recall the group structure of the fibers. Each good fiber $X_t$, $t\in S$, becomes a group with $\sigma_0(t)$ as the identity. For bad fibers, things are more complicated: given $s\in\Sigma$, take the union of the components of multiplicity one of $X_s$, and delete all singular points. We call this $X_s'$, and it becomes a group with $\sigma_0(s)$ as the identity ($X_s'$ is an extension of $\mathbf{C}$ or $\mathbf{C}^*$ by the finite group of components of multiplicity one – see [14, III.17]). The component of $X_s$ containing $\sigma_0(s)$ is called the zero component.

Then $\bar{X}'=X\cup\bigcup_{s\in\Sigma} X_s'$ is a commutative group variety over $\bar{S}$ with $\sigma_0$ as zero section. Since any section of $\bar{f}\colon \bar{X}\to\bar{S}$ lands in $\bar{X}'$, the set $\mathfrak{S}$ of all sections of $\bar{f}$ is a group with $\sigma_0$ as the identity. Since the $j$-invariant of $\bar{f}$ is non-constant, $\mathfrak{S}$ is finitely generated (this is the Mordell-Weil Theorem). Let $\mathfrak{S}_0$ be the subgroup consisting of those sections which hit the zero component of $X_s$ for $s\in\Sigma$.

The above notation will be used throughout the paper.

*A*. We first compute the map $R^1\bar{f}_*\mathbf{Q}\to j_*R^1f_*\mathbf{Q}$:

(1.2) **Lemma.** *The map* $R^1\bar{f}_*\mathbf{Q}\to j_*R^1f_*\mathbf{Q}+j_*V_{\mathbf{Q}}$ *is an isomorphism.*

*Proof*. Let $T$ be the local monodromy transformation for $s\in\Sigma$. Then the map $(R^1\bar{f}_*\mathbf{Q})_s\to(j_*R^1f_*\mathbf{Q})_s$ becomes a map:

(1.3) $H^1(X_s,\mathbf{Q})\to H^1(X_t,\mathbf{Q})^T$

where $X_t$ is a good fiber for some $t$ near $s$. Using [8, §6 and §9], one sees that the groups in (1.3) are either both 0 or both $\mathbf{Q}$. Since (1.3) is a surjection by the local invariant cycle theorem (see [1]), it must be an isomorphism. □

It is well known that $(V_{\mathbf{Z}})^{\pi_1(S)}$ is zero and $H^2(\bar{S},j_*R^1f_*\mathbf{Z})$ is finite (see [8, §11]). Thus, from (1.2), we see that $H^0(\bar{S},R^1\bar{f}_*\mathbf{Q})=H^2(\bar{S},R^1\bar{f}_*\mathbf{Q})=0$. From this we easily see that all differentials in the Leray spectral sequence for $\bar{f}$ over $\mathbf{Q}$ vanish. Hence:

(1.4) **Lemma.** *The Leray spectral sequence of* $\bar{f}\colon \bar{X}\to\bar{S}$ *degenerates at* $E_2$ *over* $\mathbf{Q}$. □

This is also true for quite general reasons – see [25, §15].

Let us make some remarks about the situation over $\mathbf{Z}$. With a little care, one can improve (1.2) to show that $R^1\bar{f}_*\mathbf{Z}\to j_*V_{\mathbf{Z}}$ is an isomorphism. Then $H^2(\bar{S},R^1\bar{f}_*\mathbf{Z})$ is finite, yet $H^3(X,\mathbf{Z})$ is torsion-free by (1.48). From this one easily proves that the differential:

(1.5) $d_2^{0,2}\colon H^0(\bar{S},R^2\bar{f}_*\mathbf{Z})\to H^2(\bar{S},R^1\bar{f}_*\mathbf{Z})$

is surjective, and that the Leray spectral sequence degenerates at $\mathrm{E}_2$ over $\mathbf{Z}$ if and only if $H^2(\bar{S},R^1\bar{f}_*\mathbf{Z})=0$. This is interesting in light of (1.30).

One can also prove that $H^1(\bar{S},R^1\bar{f}_*\mathbf{Z})$ is torsion-free, a result of [23].

*B*. We need some notation. The Leray filtration on $H^2(\bar{X},\mathbf{Q})$ is $L^2\subseteq L^1\subseteq L^0=H^2(\bar{X},\mathbf{Q})$, where, by (1.4):

$$L^1=\ker(H^2(\bar{X},\mathbf{Q})\to H^0(\bar{S},R^2\bar{f}_*\mathbf{Q})),$$
$$L^1/L^2\simeq H^1(\bar{S},R^1\bar{f}_*\mathbf{Q}),$$
$$L^2=\mathrm{im}(H^2(\bar{S},\mathbf{Q})\to H^2(\bar{X},\mathbf{Q}))=\mathbf{Q}\cdot[X_t]\qquad(t\in S).$$

Also, for $s\in\Sigma$, we write $X_s=\sum_{i\geq 0} m_i^s C_i^s$, where we label the $C_i^s$ so that $C_0^s$ is the zero component of $X_s$.

(1.6) **Theorem.** *Let* $\sigma$ *be in* $\mathfrak{S}$. *Then:*

1. *There is a rational linear combination* $\sum_{s\in\Sigma} D_s$ *of the components of bad fibers* ($D_s=\sum_i a_i^s C_i^s$, $a_i^s\in\mathbf{Q}$) *so that* $[\sigma-\sigma_0+\sum_s D_s]$ *lies in* $L^1$.

2. *The cohomology class* $[\sigma-\sigma_0+\sum_{s\in\Sigma} D_s]$ *gives a well-defined element* $\delta(\sigma)$ *of* $H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})$, *and the* map $\delta\colon \mathfrak{S}\to H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})$ *is a homomorphism.*

3. *Each* $D_s$, $s\in\Sigma$, *is unique up to a rational multiple of* $X_s$, *and is computed as follows. Assume that* $\sigma$ *hits* $C_k^s$. *If* $k=0$, *we can choose* $D_s=0$; *if* $k\neq 0$, *then* $D_s$ *satisfies the equations:*

$$(1.7)\quad D_s\cdot C_i^s=\begin{cases} 1 & i=0\\ -1 & i=k\\ 0 & \textit{otherwise.}\end{cases}$$

*Proof.* $L^1$ consists of those elements which restrict to zero in every fiber. For any irreducible curve $C$ on $\bar{X}$, there is a commutative diagram:

$$(1.8)\quad \begin{tikzcd} H^2(\bar{X},\mathbf{Q}) \arrow[rr] \arrow[dr] & & \mathbf{Q} \\ & H^2(C,\mathbf{Q}). \arrow[ur, "\wr"'] & \end{tikzcd}$$

Then one easily shows that $a\in L^1$ if and only if $a\cdot C_i^s=0$ for all $s$ and $i$. Thus, the assertion that $[\sigma-\sigma_0+\sum_s D]\in L^1$ is equivalent to the statement:

$$(1.9)\quad (\sigma_0-\sigma)\cdot C_i^s=D_s\cdot C_i^s \quad \text{for all } s \text{ and } i.$$

Note that $D_s\cdot X_s=(\sigma_0-\sigma)\cdot X_s=0$, so that

$$D_s\cdot C_0^s=-\sum_{i>0} m_i^s D_s\cdot C_i^s \quad\text{and}\quad (\sigma_0-\sigma)\cdot C_0^s=-\sum_{i>0} m_i^s(\sigma_0-\sigma)\cdot C_i^s.$$

If we set $D_s'=D_s-a_0^s X_s=\sum_{i>0} b_i^s C_i^s$, we then see that (1.9) is equivalent to:

$$(1.10)\quad (\sigma_0-\sigma)\cdot C_i^s=D_s'\cdot C_i^s \quad \text{for all } s \text{ and } i>0.$$

Since the matrix $(C_i^s\cdot C_j^s)_{i,j>0}$ is negative definite (see [23, Lemma 1.3]), (1.10) has a unique solutions. When $k=0$, $D_s'=0$, so $D_s$ is a multiple of $X_s$, and when $k\neq 0$, the unique solution $D_s'$ of (1.10) gives a solution $D_s$ of (1.7) that is unique up to a multiple of $X_s$.

Since $L^2=\mathbf{Q}\cdot[X_t]$ and $[X_t]=[X_s]$, the uniqueness above shows that $\delta(\sigma)=[\sigma-\sigma_0+\sum_{s\in\Sigma} D_s]$ is a well-defined element of $L^1/L^2=H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})=H^1(\bar{S}, j_*V_{\mathbf{Q}})$. To show that $\delta$ is a homomorphism, we need only check that the map:

$$(1.11)\quad \mathfrak{S}\to H^1(\bar{S}, j_*V_{\mathbf{Q}})\to H^1(S, V_{\mathbf{Q}})$$

is a homomorphism. (The Leray spectral sequence of $j$ yields that $H^1(\bar{S}, j_*V_{\mathbf{Q}})\to H^1(S, V_{\mathbf{Q}})$ is injective.) The map (1.11) clearly sends $\sigma$ to the cohomology class $[\sigma-\sigma_0]$, and the proof of Proposition 3.9 of [24] shows that this map is a homomorphism. □

(1.12) *Remark.* In § 3 we observe that $\delta(\sigma)$ lies in the (1, 1) part of the natural Hodge structure on $H^1(\bar{S}, R^1\bar{f}_*\mathbf{C}) = H^1(\bar{S}, j_* V)$ constructed in [25].

Take $\sigma \in \mathfrak{S}$. We can normalize the $D_s$ described in (1.6) so that $D_s \cdot \sigma_0 = 0$. This defines $D_s$ uniquely, so we write it $D_s(\sigma)$. Since $D_s(\sigma)$ is actually determined by the component (necessarily of multiplicity one) of $X_s$ hit by $\sigma$, it is easy to compute all of the possibilities.

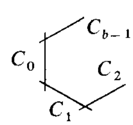
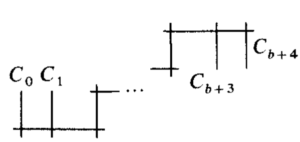
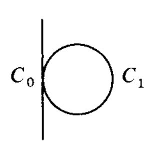
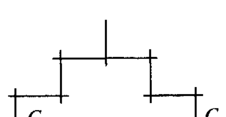
We first need to explicitly describe the bad fibers $X_s$. We use Kodaira's classification [8, § 6] (see (1.13) below).

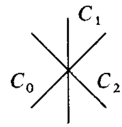
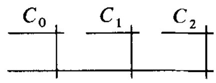
We have labeled only the components of multiplicity one; they are all we need for computations (see (1.19) and § 2). We also drop the superscript $s$, and fiber types II and II* have been omitted because they both have only $C_0$ as a component of multiplicity one.

From (1.13) it is easy to find $D_s(\sigma)$ which satisfies (1.7) and $D_s(\sigma) \cdot \sigma_0 = 0$. The results are listed in (1.14) below.

*C.* The homomorphism $\delta$ of (1.6) will enable us to use cup product on $H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})$ to get a pairing $\langle\ ,\ \rangle$ on $\mathfrak{S}$. We will see below that $\langle\ ,\ \rangle$ has several nice properties, and in § 2 it will play a crucial role (via (1.26)) in determining when we have generators (modulo torsion) of the group of sections of an elliptic surface with $p_g = 0$.

(1.13) Structure of the bad fibers $X_s$

| Type | Structure | Picture |
|---|---|---|
| $\mathrm{I}_b$ $(b>0)$ | $C_0 + C_1 + \ldots + C_{b-1}$ |  |
| $\mathrm{I}_b^*$ $(b \geqq 0)$ | $C_0 + C_1 + 2C_2 + \ldots + 2C_{b+2} + C_{b+3} + C_{b+4}$ |  |
| III | $C_0 + C_1$ |  |
| III* | $C_0 + C_1 + 2C_2 + 2C_3 + 3C_4 + 3C_5 + 4C_6 + 2C_7$ |  |

| | | |
|---|---|---|
| IV | $C_0+C_1+C_2$ |  |
| IV* | $C_0+C_1+C_2$ $+3C_3+2C_4+2C_5+2C_6$ |  |

(1.14) Table for finding $D_s(\sigma)$

| Type of $X_s$ | Component hit by $\sigma$ | $D_s(\sigma)$ |
|---|---|---|
| Arbitrary | $C_0$ | 0 |
| $I_b$ | $C_k$ $0<k<b$ | $(b-k)/b\cdot C_1+2(b-k)/b\cdot C_2+\ldots+k(b-k)/b\cdot C_k$ $+k(b-k-1)/b\cdot C_{k+1}+\ldots+k/b\cdot C_{b-1}$ |
| $I_b^*$ | $C_1$ | $C_1+C_2+\ldots+C_{b+2}+1/2\cdot C_{b+3}+1/2\cdot C_{b+4}$ |
| | $C_{b+3}$ | $1/2\cdot C_1+2/2\cdot C_2+3/2\cdot C_3+\ldots+(b+2)/2\cdot C_{b+2}+(b+4)/4\cdot C_{b+3}$ $+(b+2)/4\cdot C_{b+4}$ |
| | $C_{b+4}$ | $1/2\cdot C_1+2/2\cdot C_2+3/2\cdot C_3+\ldots+(b+2)/2\cdot C_{b+2}+(b+2)/4\cdot C_{b+3}$ $+(b+4)/4\cdot C_{b+4}$ |
| III | $C_1$ | $1/2\cdot C_1$ |
| III* | $C_1$ | $3/2\cdot C_1+C_2+3/2\cdot C_3+2C_4+5/2\cdot C_5+3C_6+3/2\cdot C_7$ |
| IV | $C_1$ | $2/3\cdot C_1+1/3\cdot C_2$ |
| | $C_2$ | $1/3\cdot C_1+2/3\cdot C_2$ |
| IV* | $C_1$ | $4/3\cdot C_1+2/3\cdot C_2+C_3+5/3\cdot C_4+4/3\cdot C_5+2C_6$ |
| | $C_2$ | $2/3\cdot C_1+4/3\cdot C_2+C_3+4/3\cdot C_4+5/3\cdot C_5+2C_6$ |

The whole Leray spectral sequence for $\bar{f}\colon \bar{X}\to\bar{S}$ has cup products; in particular, there is a cup product

$$\cup\colon H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})\otimes H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})\to\mathbf{Q}$$

compatible with the usual cup product on $H^2(\bar{\mathrm{X}}, \mathbf{Q})$. Then, for $\sigma$ and $\sigma'$ in $\mathfrak{S}$, we define:

$$\langle\sigma,\sigma'\rangle=-(\delta(\sigma)\cup\delta(\sigma')).$$

(1.15) **Lemma.** *$\langle\ ,\ \rangle$ is a bilinear form on $\mathfrak{S}$. Furthermore, for $\sigma\in\mathfrak{S}$, $\langle\sigma,\sigma\rangle\geqq 0$, and $\langle\sigma,\sigma\rangle=0$ if and only if $\sigma$ is torsion.*

*Proof.* $\langle\ ,\ \rangle$ is bilinear because $\delta$ is a homomorphism. If $\sigma$ is in $\mathfrak{S}$, choose an integer $n$ so that $n\sigma$ is contained in $\mathfrak{S}_0$, i.e., $n\sigma$ hits the zero component of $X_s$ for all $s\in\Sigma$. Then (1.14) show that $\delta(n\sigma)=[n\sigma-\sigma_0]$. Thus

$$n^2\langle\sigma,\sigma\rangle=\langle n\sigma,n\sigma\rangle=-(n\sigma)^2+2(n\sigma)\cdot\sigma_0-\sigma_0^2.$$

But all sections have the same self-intersection $\sigma_0^2$, which is a negative number by (2.4). So $n^2\langle\sigma,\sigma\rangle=-2\sigma_0^2+2(n\sigma)\cdot\sigma_0$. If $\sigma$ is not torsion, then $n\sigma$ and $\sigma_0$ are distinct divisors, so that $(n\sigma)\cdot\sigma_0\geqq 0$, which implies $n^2\langle\sigma,\sigma\rangle>0$. The lemma follows. $\square$

(1.16) **Corollary.** *$\delta$: $\mathfrak{S}\to H^1(\bar{S},R^1\bar{f}_*\mathbf{Q})$ is injective modulo torsion.* $\square$

The proof of (1.15) also yields the following known fact (see [23, Prop. 1.6]):

(1.17) **Corollary.** *$\mathfrak{S}_0$ is torsion-free.* $\square$

Since cup product is negative definite on the (1, 1) part of the Hodge structure on $H^1(\bar{S},j_*V)$, we get another proof (via (1.12)) that $\langle\sigma,\sigma\rangle\geqq 0$ for $\sigma\in\mathfrak{S}$.

The first step in computing $\langle\sigma,\sigma'\rangle$ is:

(1.18) **Lemma.** *For $\sigma,\sigma'$ in $\mathfrak{S}$, $\sigma\cdot D_s(\sigma')=\sigma'\cdot D_s(\sigma)$, and*

$$\langle\sigma,\sigma'\rangle=-(\sigma-\sigma_0)\cdot(\sigma'-\sigma_0)-\sum_{s\in\Sigma}\sigma\cdot D_s(\sigma').$$

*Proof.* Suppose that $\sigma$ hits $C_k^s$, and write $D_s(\sigma')=\sum_{i>0}a_i\cdot C_i^s$. Then $\sigma\cdot D_s(\sigma')=a_k$, and from (1.7) we see that $D_s(\sigma)\cdot D_s(\sigma')=-a_k$. The first equality of the lemma then follows by symmetry, and the second is now an easy computation. $\square$

Thus $\langle\sigma,\sigma'\rangle$ has a "geometric part," $-(\sigma-\sigma_0)\cdot(\sigma'-\sigma_0)$, and then "correction terms" coming from the behavior of $\sigma$ and $\sigma'$ at the bad fibers. Using (1.14), it is easy to make a table giving all the correction terms:

(1.19) Table of the local correction terms $\sigma\cdot D_s(\sigma')$

| Type of $X_s$ | Criterion (see (1.13)) | Correction Factor |
|---|---|---|
| Arbitrary | $\sigma$ or $\sigma'$ hits the zero component | 0 |
| $\mathrm{I}_b$ | $\sigma$ hits $C_k$, $\sigma'$ hits $C_{k'}$, $0<k\leqq k'$ | $k(b-k')/b$ |
| $\mathrm{I}_b^*$ | $\sigma,\sigma'$ hit $C_1$ | 1 |
| | $\sigma,\sigma'$ both hit $C_{b+3}$ or $C_{b+4}$ | $(b+4)/4$ |
| | One hits $C_{b+3}$, the other $C_{b+4}$ | $(b+2)/2$ |
| | One hits $C_1$, the other $C_{b+3}$ or $C_{b+4}$ | 1/2 |
| III | $\sigma,\sigma'$ hit $C_1$ | 1/2 |
| III* | $\sigma,\sigma'$ hit $C_1$ | 3/2 |

| | | |
|---|---|---|
| IV | $\sigma, \sigma'$ both hit $C_1$ or $C_2$ | 2/3 |
| | One hits $C_1$, the other hits $C_2$ | 1/3 |
| IV* | $\sigma, \sigma'$ both hit $C_1$ or $C_2$ | 4/3 |
| | One hits $C_1$, the other hits $C_2$ | 2/3 |

Note that the greatest denominator that can occur is the *exponent* of the group $G_s$ of components of $X_s$ of multiplicity one (see [14, III.17] – type $\mathrm{I}_b^*$ is especially interesting). This tells us the following:

(1.20) **Corollary.** *Let $N$ be the l.c.m. of the exponents of the groups of components of multiplicity one of the bad fibers $X_s$, $s \in \Sigma$. Then $N\langle\ ,\ \rangle \in \mathbf{Z}$.* □

Finally, we want to compare $\langle\ ,\ \rangle$ on $\mathfrak{S}$ to the usual intersection form $(\ ,\ )$ on $NS(\bar{X})$, the Néron-Severi group of $\bar{X}$ (note that by (1.48), $NS(\bar{X})$ is torsion-free). Forms like these have a discriminant defined as follows. Let $(\ ,\ )$ be a $\mathbf{Q}$-valued form on a finitely generated abelian group $G$ of rank $r$. If $\sigma_1, \ldots, \sigma_r$ generate $G$ modulo torsion, then define:

(1.21) $\operatorname{disc}(\ ,\ )_G = \det(\sigma_i, \sigma_j)/(\# G_{\mathrm{tor}})^2$

where $G_{\mathrm{tor}}$ is the torsion subgroup of $G$ (and the determinant of a $0 \times 0$ matrix is 1). Then we have:

(1.22) **Proposition.** *Let $m_s$ be the number of components of multiplicity one in the fiber $X_s$. Then:*

$$\operatorname{disc}\langle\ ,\ \rangle_{\mathfrak{S}} = |\operatorname{disc}(\ ,\ )_{NS(\bar{X})}| / \prod_{s \in \Sigma} m_s.$$

*Proof.* If $\sigma = \sum b_i \sigma_i$ in $\mathfrak{S}$, then the divisors $\sigma - \sigma_0$ and $\sum b_i(\sigma_i - \sigma_0)$ are linearly equivalent on the generic fiber of $\bar{f}$ (Abel's theorem). Thus, in $NS(\bar{X})$ we have:

(1.23) $$\sigma - \sigma_0 = \sum b_i(\sigma_i - \sigma_0) + a X_t + \sum_{i>0} b_i^s C_i^s$$

where $X_t$ and the $C_i^s$ are as in (1.6). From this and Theorem 1.1 of [23], we see that the map sending $\sigma$ to $\sigma - \sigma_0$ gives an isomorphism:

(1.24) $$\mathfrak{S} \simeq NS(\bar{X})/(\mathbf{Z}[\sigma_0] + \mathbf{Z}[F] + \sum_{i>0} \mathbf{Z}[C_i^s]).$$

Let $H$ be the subgroup of $NS(\bar{X})$ generated by the classes of $\sigma_0, X_t, C_i^s$ (for $i>0$) and $\sigma$ (for $\sigma \in \mathfrak{S}_0$). If $\sigma_1, \ldots, \sigma_r$ is a basis of $\mathfrak{S}_0$, then (1.23) shows that $H$ is spanned by $[\sigma_0]$, $[X_t]$, $[C_i^s]$ $(i>0)$ and $a_j = [\sigma_j - \sigma_0 - ((\sigma_j - \sigma_0)\cdot\sigma_0) X_t]$ (note that $a_j \cdot C_i^s = a_j \cdot X_t = a_j \cdot \sigma_0 = 0$). Then one easily computes that:

(1.25) $$|\operatorname{disc}(\ ,\ )_H| = \operatorname{disc}\langle\ ,\ \rangle_{\mathfrak{S}_0} \cdot \prod_s m_s$$

because $\det(a_i, a_j) = \det\langle\sigma_i, \sigma_j\rangle$ and $\det(C_i^s \cdot C_j^s) = m_s$ (see Lemma 1.3 in [23]).

Let $G_s$ be the group of components of multiplicity one in $X_s$. The natural map $\mathfrak{S} \to \bigoplus G_s$ (evaluating which components a section hits) gives us, via (1.24), a homomorphism $NS(\bar{X}) \to \bigoplus G_s$. This kernel of this map is $H$ (use (1.24)), so that $[NS(\bar{X}): H] = [\mathfrak{S}: \mathfrak{S}_0]$, and then (1.25) and Lemma 1.8 of [23] give us:

$$\begin{aligned}\operatorname{disc}\langle\ ,\ \rangle_{\mathfrak{S}} &= \operatorname{disc}\langle\ ,\ \rangle_{\mathfrak{S}_0}/[\mathfrak{S}: \mathfrak{S}_0]^2 \\ &= |\operatorname{disc}(\ ,\ )_H|/[\mathfrak{S}: \mathfrak{S}_0]^2 \cdot \prod_{s\in\Sigma} m_s \\ &= |\operatorname{disc}(\ ,\ )_{NS(\bar{X})}|/\prod_{s\in\Sigma} m_s. \quad \square\end{aligned}$$

This proposition is proved (under a very restrictive hypothesis) in [23, Corollary 1.7].

If $p_g(\bar{X}) = 0$, then $NS(\bar{X}) = H^2(\bar{X}, \mathbf{Z})$ and $(\ ,\ )$ has discriminant 1 by Poincaré duality. From (1.21) and (1.22) we then get:

(1.26) **Corollary.** *If* $p_g(\bar{X}) = 0$, *then* $\operatorname{disc}\langle\ ,\ \rangle_{\mathfrak{S}} = 1/\prod_{s\in\Sigma} m_s$, *and this means the following:*

1. *If* $\mathfrak{S}$ *has rank* 0, *then* $(\#\mathfrak{S})^2 = \prod_{s\in\Sigma} m_s$.

2. *If* $\mathfrak{S}$ *has rank* $r > 0$, *then* $\sigma_1, \ldots, \sigma_r$ *in* $\mathfrak{S}$ *generate modulo torsion if and only if:*

$$\det\langle\sigma_i, \sigma_j\rangle = (\#\mathfrak{S}_{\text{tor}})^2/\prod_{s\in\Sigma} m_s. \quad \square$$

We can also compute the discriminant of cup product on $H^1(\bar{S}, R^1\bar{f}_*\mathbf{Z})$:

(1.27) **Proposition.**

$$\operatorname{disc}(\ ,\ )_{H^1(\bar{S}, R^1\bar{f}_*\mathbf{Z})} = \left(\prod_{s\in\Sigma} m_s\right)/(\# H^2(\bar{S}, R^1\bar{f}_*\mathbf{Z}))^2.$$

*Proof.* We will just sketch the proof. Let $L^1$ be the kernel of $\pi$: $H^2(\bar{X}, \mathbf{Z}) \to H^0(\bar{S}, R^2\bar{f}_*\mathbf{Z})$, so that $L^1/\mathbf{Z}[X_t] \simeq H^1(\bar{S}, R^1\bar{f}_*\mathbf{Z})$. Let $N$ be generated by $L^1$, $\sigma_0$ and the $C_i^s$. Then the proof of (1.22) (especially (1.25)) shows that:

$$|\operatorname{disc}(\ ,\ )_N| = |\operatorname{disc}(\ ,\ )_{H^1(\bar{S}, R^1\bar{f}_*\mathbf{Z})}| \cdot \prod_{s\in\Sigma} m_s.$$

Thus, we need to determine the index of $N$ in $H^2(\bar{X}, \mathbf{Z})$, which is the same as the index of their images under the map $\pi$ above.

Using (1.8) and Lemma 1.3 of [23], one easily sees that the image of $N$ in $H^2(X_s, \mathbf{Z})$ has index $m_s$, so that $\pi(N)$ has index $\prod_{s\in\Sigma} m_s$ in $H^0(\bar{S}, R^2\bar{f}_*\mathbf{Z})$. Since $\pi(H^2(\bar{X}, \mathbf{Z}))$ has index $\# H^2(\bar{S}, R^1\bar{f}_*\mathbf{Z})$ (this is (1.5)), we are done. $\square$

(1.28) **Corollary.** $(\# H^2(\bar{S}, R^1\bar{f}_*\mathbf{Z}))^2 \,\Big|\, \prod_{s\in\Sigma} m_s$. $\square$

We can say some more about the relation of $\# H^2(\bar{S}, R^1\bar{f}_*\mathbf{Z})$ to other invariants.

(1.29) **Proposition.**

$$[\mathfrak{S}:\mathfrak{S}_0]\cdot(\#H^2(\bar{S},R^1\bar{f}_*\mathbf{Z}))\Big|\prod_{s\in\Sigma}m_s.$$

*If* $p_g(\bar{X})=0$, *then:*

1. $[\mathfrak{S}:\mathfrak{S}_0]\cdot(\#H^2(\bar{S},R^1\bar{f}_*\mathbf{Z}))=\prod_{s\in\Sigma}m_s$.
2. $\#H^2(\bar{S},R^1\bar{f}_*\mathbf{Z})\Big|[\mathfrak{S}:\mathfrak{S}_0]$.
3. *If* $\mathfrak{S}$ *is finite, then*

$$\#\mathfrak{S}=\#H^2(\bar{S},R^1\bar{f}_*\mathbf{Z})=\Big(\prod_{s\in\Sigma}m_s\Big)^{1/2}.$$

*Proof.* Since $\pi(N)\subseteq\pi(NS(\bar{X}))\subseteq\pi(H^2(\bar{X},\mathbf{Z}))$, we see that

$$[\pi(NS(\bar{X})):\pi(N)]\cdot(\#H^2(\bar{S},R^1\bar{f}_*\mathbf{Z}))\Big|\prod_{s\in\Sigma}m_s,$$

with equality when $p_g=0$. If $H$ is as in the proof of (1.22) (where we discovered that $[NS(\bar{X}):H]=[\mathfrak{S}:\mathfrak{S}_0]$), then we see that $\pi(N)=\pi(H)$. Thus, we need only prove that the kernel of $NS(\bar{X})\to H^0(\bar{S},R^2\bar{f}_*\mathbf{Z})$ lies in $H$. This follows from (1.6), (1.24) and that fact that $D_s(\sigma)$ has integral coefficients only if it is zero (see (1.14)). □

The following is also true:

(1.30) **Proposition.** *There are natural isomorphisms:*

$$\mathfrak{S}_{\mathrm{tor}}\simeq H^1(S,R^1f_*\mathbf{Z})_{\mathrm{tor}}\simeq\mathrm{Hom}(H^2(\bar{S},R^1\bar{f}_*\mathbf{Z}),\mathbf{Q}/\mathbf{Z}).$$

*Thus* $\#\mathfrak{S}_{\mathrm{tor}}=\#H^2(\bar{S},R^1\bar{f}_*\mathbf{Z})$ (cf. (1.27)–(1.29)).

[We had originally conjectured a version of (1.30) for the case $p_g=0$. Subsequently, P. Deligne showed that (1.30) is true in general, and A. Kas later gave a different proof. The most natural proof (not given here) uses the isomorphism

$$\mathfrak{S}_{\mathrm{tor}}\xrightarrow{\sim}H^1(S,R^1f_*\mathbf{Z})_{\mathrm{tor}}$$

and Poincaré duality.]

*D.* Let $E$ be the generic fiber of $\bar{f}:\bar{X}\to\bar{S}$ (so that $\mathfrak{S}=E(K)$, where $K=K(\bar{S})$), and let $D$ be a divisor on $E$. Then, as described in [13], there is a height function $h_D$: $\mathfrak{S}\to\mathbf{Z}$ which measures the "size" of rational points on $E$. Since $E$ is an abelian variety over $K$, there is a unique quadratic function $\hat{h}_D$: $\mathfrak{S}\to\mathbf{R}$ (the Tate height relative to $D$ – see [13]) such that:

(1.31) $h_D=\hat{h}_D+O(1)$.

$\hat{h}_D$ being quadratic means that we can write it as

$$\hat{h}_D(\sigma)=(1/2)\,f(\sigma,\sigma)+\ell(\sigma),$$

where $f$ (resp. $\ell$) is bilinear (resp. linear) on $\mathfrak{S}$.

We want to describe $\hat{h}_D$ in terms of $\langle\ ,\ \rangle$. We first lift $\delta\colon \mathfrak{S}\to H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})$ of (1.6) to a homomorphism mapping into $H^2(\bar{X}, \mathbf{Q})$:

(1.32) **Lemma.** *The map* $\delta\colon \mathfrak{S}\to H^2(\bar{X}, \mathbf{Q})$ *defined by*

$$\delta(\sigma)=[\sigma-\sigma_0+\sum_s D_s(\sigma)-((\sigma-\sigma_0)\cdot\sigma_0)\,X_t]$$

*($X_t$ is a good fiber) is a homomorphism.*

*Proof.* $\delta(\sigma)$, as defined above, is the unique element of $L^1$ which satisfies $\delta(\sigma)\cdot\sigma_0=0$ and reduces to the $\delta(\sigma)$ of (1.14) in $H^1(\bar{S}, R^1\bar{f}_*\mathbf{Q})$. It follows easily that $\delta$ is a homomorphism. $\square$

The divisor $D$ on $E$ gives a unique divisor $\bar{D}$ on $\bar{X}$ which contains no component of any fiber and pulls back to $D$ on $E$. Then the Tate height $h_D$ can be expressed in terms of $\bar{D}$, $\langle\ ,\ \rangle$, and $\delta$ as follows:

(1.33) **Theorem.** *For* $\sigma\in\mathfrak{S}$, *we have:*

$$\begin{aligned}\hat{h}_D(\sigma)&=(1/2)\langle\sigma,\sigma\rangle\deg D+(\bar{D}\cdot\delta(\sigma))\\&=\bar{D}\cdot(\sigma-\sigma_0+\sum_s D_s(\sigma)-(1/2)(\sum_s\sigma\cdot D_s(\sigma))X_t).\end{aligned}$$

*Proof.* The two formulas on the right hand side are equal by (1.32) and (1.18) (note that $\deg D=\bar{D}\cdot X_t$), and they define a quadratic function which we call $g(\sigma)$. Since we have

$$h_D(\sigma)=\bar{D}\cdot\sigma$$

(see [13, Theorem 4]), we get the formula:

$$h_D(\sigma)-g(\sigma)=\deg D-\sum_s\bar{D}\cdot D_s(\sigma)+1/2(\sum_s\sigma\cdot D_s(\sigma))\deg D.$$

Using (1.14) and (1.19), it is easy to find a constant $C$ (depending only on $\bar{D}$ and the bad fibers) such that

$$|h_D(\sigma)-g(\sigma)|\leqq C$$

for all $\sigma\in\mathfrak{S}$. Since $\hat{h}_D(\sigma)$ is the unique quadratic function with this property, we must have $\hat{h}_D=g$. $\square$

We can use this to strengthen a result of [13]:

(1.34) **Corollary.** *The following are equivalent:*

1. $\sigma\in\mathfrak{S}_0$.
2. *For every divisor $D$ on $E$,* $\hat{h}_D(\sigma)=\bar{D}\cdot(\sigma-\sigma_0)$.

*Proof.* $1\Rightarrow 2$. If $\sigma\in\mathfrak{S}_0$, then $D_s(\sigma)=0$ for every $s\in\Sigma$, and the second statement follows from (1.33).

$2\Rightarrow 1$. Let $D=\sigma$. Then we get

$$\hat{h}_D(\sigma)-\bar{D}\cdot(\sigma-\sigma_0)=(1/2)\sum_s\sigma\cdot D_s(\sigma),$$

and (1.19) shows that the left hand side of the above is zero only when $\sigma \in \mathfrak{S}_0$. □

We can also describe how our methods compare to those used by Néron in [15]. The basic tool used in [15] is a pairing $(D, \mathfrak{A})$ defined for divisors $D$ and $\mathfrak{A}$ on $E$, where $\deg \mathfrak{A} = 0$. This gives a bilinear pairing:

$$\langle \sigma, \sigma' \rangle_D = (D_\sigma - D, \sigma' - \sigma_0).$$

(1.35) **Proposition.** *Let $\sigma, \sigma'$ be in $\mathfrak{S}$. Then:*
1. $(D, \sigma - \sigma_0) = -(1/2)\langle \sigma, \sigma \rangle \deg D - (\bar{D} \cdot \delta(\sigma))$,
2. $\langle \sigma, \sigma' \rangle_D = -\langle \sigma, \sigma' \rangle \deg D$.

*Proof.* This follows from (1.33) and Proposition 11 of [15, II.14] (where Néron shows that $\hat{h}_D(\sigma) = -(D, \sigma - \sigma_0)$ with $-(1/2)\langle \sigma, \sigma \rangle_D$ as its bilinear part). □

From (1.33) and (1.35) we get

$$(D, \sigma - \sigma_0) = -\bar{D} \cdot (\sigma - \sigma_0) - \sum_s [(\bar{D} \cdot D_s(\sigma)) - (1/2)(\sigma \cdot D_s(\sigma)) \deg D],$$

which, when compared to the formula (see [15, III.6])

$$(D, \sigma - \sigma_0) = -\bar{D} \cdot (\sigma - \sigma_0) + \sum_s j_s(D, \sigma - \sigma_0),$$

leads us to conjecture that, for $s \in \Sigma$, we have:

$$j_s(D, \sigma - \sigma_0) = -(\bar{D} \cdot D_s(\sigma)) + (1/2)(\sigma \cdot D_s(\sigma)) \deg D.$$

## Appendix to § 1

### *The First Homotopy and Homology Groups of an Elliptic Surface*

Let $\bar{f}: \bar{X} \to \bar{S}$ be an arbitrary elliptic fibration (not necessarily algebraic). Then $\bar{f}$ induces maps $\bar{f}_*: \pi_1(\bar{X}) \to \pi_1(\bar{S})$ and $\bar{f}_*: H_1(\bar{X}, \mathbf{Z}) \to H_1(\bar{S}, \mathbf{Z})$, and we want to know when these maps are isomorphisms. For the fundamental group, we give necessary and sufficient conditions for this to be true (see (1.36)). For homology, our results are not as complete (see (1.40), (1.44) and (1.47)).

We will say that $\bar{f}$ has non-trivial local monodromy if there is $s \in \Sigma$ so that $R^1 \bar{f}_* \mathbf{Z}$ is non-constant in a neighborhood of $s$. This is equivalent to the minimal model of $\bar{f}$ having at least one fiber not of type ${}_mI_0$, $m \geqq 0$. If the $j$-invariant of $\bar{f}$ is non-constant, this condition is certainly satisfied.

(1.36) **Proposition.** *Let $\bar{f}: \bar{X} \to \bar{S}$ be an elliptic fibration, and let $\bar{S}$ have genus $g$. The following are equivalent:*
1. *$\bar{f}_*: \pi_1(\bar{X}) \to \pi_1(\bar{S})$ is an isomorphism.*
2. *$\bar{f}$ has non-trivial local monodromy, and:*

a) *If $g \geqq 1$, then $\bar{f}$ has no multiple singular fibers.*

b) *If $g = 0$, then $\bar{f}$ has $\leqq 2$ multiple singular fibers, and if it does have 2, then their multiplicities are relatively prime.*

*Proof.* $1 \Rightarrow 2$. We first consider the multiple singular fibers of $\bar{f}$. We can assume that $\bar{f}$ is minimal (this does not affect $\pi_1$). If $g = 0$, then $\pi_1(\bar{X}) = 0$, and we are done by the proof of Proposition 2 in [9]. If $g \geqq 1$, we will show that the existence of one multiple singular fiber $X_s$ (of multiplicity $m$) leads to a contradiction.

Let $S_0 = \bar{S} - \{s\}$. Because $g \geqq 1$, we can find a finite ramified normal covering $\bar{S}' \to \bar{S}$ with group $G$, unramified over $S_0$, where every preimage of $s$ has ramification index $m$. If $\bar{X}'$ is the normalization of $\bar{X} \times_{\bar{S}} \bar{S}'$, then $\bar{X}' \to \bar{S}'$ is an elliptic fibration, but more importantly, the map $\bar{X}' \to \bar{X}$ is a covering space with group $G$ (see [9, § 1]).

The isomorphism $\pi_1(\bar{X}) \xrightarrow{\sim} \pi_1(\bar{S})$ means that every covering space of $\bar{X}$ is the pull-back of a covering space of $\bar{S}$. Yet the covering $\bar{X}' \to \bar{X}$ constructed above clearly cannot arise in this manner. Thus, we reach a contradiction.

Next, assume that $\bar{f}$ has trivial local monodromy. If $g \geqq 1$, then $\bar{f}$ has no multiple singular fibers, so that $\bar{f}$ must be smooth. We then get an exact sequence:

$$1 \to \pi_1(X_t) \to \pi_1(\bar{X}) \xrightarrow{\bar{f}_*} \pi_1(\bar{S}) \to 1$$

since $\pi_2(\bar{S}) = 0$. This is impossible because $\bar{f}_*$ is an isomorphism. When $g = 0$, assume that $\bar{f}$ is minimal, so that the only bad fibers are of type ${}_mI_0$, $m > 0$. Then Theorems 6 and 7 of [20, Ch. IV] show that $\chi(\bar{X}) = 0$, so that $q = p_g + 1 \geqq 1$ by Noether's formula. Since $\bar{f}_*$ is an isomorphism, $b_1 = 0$. Then $q = 0$ by [10, Theorem 3], again giving us a contradiction.

The proof of $2 \Rightarrow 1$ is an immediate consequence of the following two lemmas.

(1.37) **Lemma.** *Let $g$ be the genus of $\bar{S}$, and assume the following:*

1. *If $g \geqq 1$, then $\bar{f}$ has no multiple singular fibers.*

2. *If $g = 0$, then $\bar{f}$ has $\leqq 2$ multiple singular fibers, and if it does have 2, then their multiplicities are relatively prime.*

*Then, for any good fiber $X_t$, we have an exact sequence:*

$$\pi_1(X_t) \to \pi_1(\bar{X}) \to \pi_1(\bar{S}) \to 1.$$

*Proof.* The bad fibers of $\bar{f}$ are $X_s$, $s \in \Sigma$ where now each one has multiplicity $m_s$, and as usual $S = \bar{S} - \Sigma$. Taking the fundamental groups of (1.1) gives a commutative diagram:

$$\begin{tikzcd}
 & & 1 & 1 & \\
1 \arrow[r] & K_3 \arrow[r] & \pi_1(\bar{X}) \arrow[r] \arrow[u] & \pi_1(\bar{S}) \arrow[r] \arrow[u] & 1 \\
 & \pi_1(X_t) \arrow[r] \arrow[u] & \pi_1(X) \arrow[r] \arrow[u] & \pi_1(S) \arrow[r] \arrow[u] & 1 \\
 & & K_2 \arrow[r] \arrow[u] & K_1 \arrow[u] & \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd} \tag{1.38}$$

where $K_1, K_2$ and $K_3$ are the appropriate kernels. The second row is exact because $f$ is a $C^\infty$-fibration, and the columns and first row are exact because $\pi_1(X)\to\pi_1(\bar{X})$ and $\pi_1(S)\to\pi_1(\bar{S})$ are onto.

Let $C$ be a component of multiplicity $n$ in $X_s$, and let $u$ be a loop around it in $X$. Then $u$ is in $K_2$, and if $g_s$ is a loop around $s$ in $S$, then $u$ maps to $g_s^n$ in $K_1$. If $\bar{f}$ has no multiple singular fibers, then we can assume that $C$ has multiplicity one, so that $u$ maps to $g_s$. Thus, $K_2\to K_1$ is onto, and then an easy diagram chase shows that $\pi_1(X_t)\to K_3$ is onto (which is precisely what we want to prove).

If $g=0$, then $K_1$ is the group generated by the $g_s$, $s\in\Sigma$, subject to the single relation $\prod_{s\in\Sigma} g_s=1$. If we have only one multiple singular fiber, say $X_s$, then the image of $K_2$ contains $g_{s'}$ for all $s'\neq s$. Since these generate $K_1$, we again conclude that $\pi_1(X_t)\to K_3$ is onto. If we have two multiple singular fibers, $X_{s_1}$ and $X_{s_2}$, then the image of $K_2$ contains $g_{s_1}^{m_{s_1}}$, $g_{s_2}^{m_{s_2}}$ and $g_{s'}$ for $s'\neq s_1$ or $s_2$. Since $m_{s_1}$ and $m_{s_2}$ are relatively prime, the image is again all of $K_1$. $\square$

(1.39) **Lemma.** *Assume that $\bar{f}$ has non-trivial local monodromy. Then the map $\pi_1(X_t)\to\pi_1(\bar{X})$ is zero for any good fiber $X_t$.*

*Proof.* First, assume that the $j$-invariant of $\bar{f}$ is constant. We can assume that $\bar{f}$ is minimal (blowing down does nor affect $\pi_1$). Then the non-trivial monodromy of $\bar{f}$ must come from a fiber $X_s$ of type $I_0^*$, II, II*, III, III*, IV or IV*, all of which are simply connected. If $\Delta$ is a small disc around $s$ in $\bar{S}$, then $\pi_1(X_t)\to\pi_1(\bar{X})$ factors:

$$\pi_1(X_t)\to\pi_1(\bar{f}^{-1}(\Delta))\to\pi_1(\bar{X}).$$

Since $\bar{f}^{-1}(\Delta)$ has the same homotopy type as $X_s$, $\pi_1(\bar{f}^{-1}(\Delta))=0$, so that $\pi_1(X_t)\to\pi_1(\bar{X})$ factors through zero.

Next, assume that the $j$-invariant is non-constant.

If $\bar{S}'\to\bar{S}$ is a map of curves, let $\bar{X}'$ be a resolution of singularities of $\bar{X}\times_{\bar{S}}\bar{S}'$. Then we have a commutative diagram:

$$\begin{array}{ccc} \bar{X}' & \longrightarrow & \bar{X} \\ {\scriptstyle \bar{f}'}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \bar{S}' & \longrightarrow & \bar{S} \end{array}$$

which shows that $\pi_1(X_t)\to\pi_1(\bar{X})$ factors through $\pi_1(X_t)\to\pi_1(\bar{X}')$. So we need only show that the latter map is zero. As we saw above, we can assume $\bar{X}'$ is minimal over $\bar{S}'$.

Find $\bar{S}'$ so that $\bar{X}'$ has no multiple singular fibers. Then $\bar{X}'\to\bar{S}'$ is a deformation of an algebraic elliptic surface since $j$ is non-constant (see [8, § 11]). This does not change the map $\pi_1(X_t)\to\pi_1(\bar{X}')$, so we can assume that $\bar{X}$ is algebraic. Pulling back further if necessary, we can assume that the generic fiber of $\bar{f}$ has an affine equation $y^2=x(x-1)(x-\tau)$, where $\tau\in K(S)$ is non-constant (because $j$ is).

If one looks at the Legendre family $y^2=x(x-1)(x-\lambda)$ over the $\lambda$-sphere, it is easy to see that the vanishing cycles $\alpha$ and $\beta$ coming from the bad fibers at $\lambda=0$

and $\lambda=1$ form a basis for $\pi_1(X_t)$. Since $\bar{X}$ is the pull-back of this family via $\tau$, $\alpha$ (resp. $\beta$) is a vanishing cycle for $\bar{X}$ at any fiber $X_s$ where $\tau(s)=0$ (resp. $\tau(s)=1$). Since the vanishing cycles of $\bar{X}$ actually vanish in $\pi_1(\bar{X})$, we see that $\pi_1(X_t)\to\pi_1(\bar{X})$ is zero. □

We next turn to homology. We will always work with **Z** coefficients, and our first result gives sufficient conditions for $\bar{f}_*$ to be an isomorphism on $H_1$:

(1.40) **Proposition.** *Let $\bar{f}\colon \bar{X}\to\bar{S}$ be an elliptic fibration with non-trivial local monodromy, and let $m_i$, $i=1,\dots,\ell$, be the multiplicities of the multiple singular fibers of $\bar{f}$. If $\ell\geqq 2$, assume that the $m_i$ are pairwise relatively prime. Then $\bar{f}_*\colon H_1(\bar{X})\to H_1(\bar{S})$ is an isomorphism.*

This is actually an immediate consequence of the more general proposition (1.41) below.

Given any collection of integers $m_i>1$, $i=1,\dots,\ell$, let $G=G(m_1,\dots,m_\ell)$ be the cokernel of the map $\mathbf{Z}\to\bigoplus_{i=1}^{\ell}\mathbf{Z}/m_i\mathbf{Z}$ (if $\ell=0$, set $G=0$). Note that $G=0$ if and only if $\ell\leqq 1$, or $\ell\geqq 2$ and for $i\neq j$, $m_i$ and $m_j$ are relatively prime (this is the Chinese Remainder Theorem).

The following seems to be well-known (see, for example, [6]).

(1.41) **Proposition.** *Assume that $\bar{f}$ has non-trivial local monodromy and let $m_i$, $i=1,\dots,\ell$, be the multiplicities of the multiple singular fibers of $\bar{f}$. Then we have an exact sequence:*

$$0\to G(m_1,\dots,m_\ell)\to H_1(\bar{X})\to H_1(\bar{S})\to 0.$$

*Proof.* This time we take the homology of (1.1):

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & M_3 \arrow[r] & H_1(\bar{X}) \arrow[r] \arrow[u] & H_1(\bar{S}) \arrow[r] \arrow[u] & 0 \\
 & H_1(X_t) \arrow[r] \arrow[u] & H_1(X) \arrow[r] \arrow[u] & H_1(S) \arrow[r] \arrow[u] & 0 \\
 & & M_2 \arrow[r] \arrow[u] & M_1 \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{1.42}$$

The $M_i$ are the appropriate kernels, and the diagram has the same exactness properties as (1.38) (the second row of (1.42) is exact by the Serre spectral sequence for $f\colon X\to S$). The map $H_1(X_t)\to M_3$ is zero by (1.39), so that we get an exact sequence:

$$M_2\to M_1\to H_1(\bar{X})\to H_1(\bar{S})\to 0.$$

$M_1$ is the abelian group generated by $g_s$, $s\in\Sigma$ (each $g_s$ is a loop about $s$) subject to the single relation $\sum_{s\in\Sigma} g_s=0$. The proof of (1.37) shows that the image of $M_2$ in $M_1$ contains the elements $m_i g_{s_i}$, $i=1,\ldots,\ell$ and $g_s$, $s\notin\{s_1,\ldots,s_\ell\}$ (this is the notation of (1.37)). These elements generate a subgroup of $M_1$ whose quotient is $G=G(m_1,\ldots,m_\ell)$. Thus we get an exact sequence:

(1.43) $G\to H_1(\bar{X})\to H_1(\bar{S})\to 0.$

Using the usual presentation of $\pi_1(S)$ with generators $\alpha_i$, $\beta_i$ and $g_s$, $s\in\Sigma$, we get a surjective homomorphism $\pi_1(S)\to G$ by sending $g_{s_i}$ to the image of the generator of $\mathbf{Z}/m_i\mathbf{Z}$ in $G$, and sending all other generators to zero. This gives us a ramified covering $\bar{S}'\to\bar{S}$ with group $G$, unramified outside $\{s_1,\ldots,s_\ell\}$, and the ramification index at points above $s_i$ divides $m_i$. If we construct $\bar{X}'$ as in the proof of (1.36), we see that $\bar{X}'\to\bar{X}$ is a covering space with group $G$. This is classified by a map $\pi_1(\bar{X})\to G$, which gives a map $H_1(\bar{X})\to G$ since $G$ is abelian. The map $G\to H_1(\bar{X})$ from (1.43), followed by this map, is the identity on $G$, so that $G\to H_1(\bar{X})$ is injective. $\square$

Here is a partial converse to (1.40):

(1.44) **Proposition.** *If $\bar{f}_*\colon H_1(\bar{X})\to H_1(\bar{S})$ is an isomorphism, then the $m_i$ are pairwise relatively prime (see (1.40)) and the monodromy is non-trivial.*

*Proof.* The proof of (1.41) shows that even if $H^1(X_t)\to H^1(\bar{X})$ is non-zero, we still have an inclusion $G\to\operatorname{Ker}(H_1(\bar{X})\to H_1(\bar{S}))$. Thus $G=0$ and the $m_i$ are pairwise relatively prime. If the monodromy is trivial, then $b_1\geqq 2g+1$ by Theorem 14.7 of [8]. But $b_1=2g$ when $\bar{f}_*$ is an isomorphism. $\square$

Note that (1.44) says nothing about *local* monodromy. This is because having non-trivial local monodromy is not a necessary condition for $\bar{f}_*$ to be an isomorphism, yet just non-trivial monodromy is not enough (i.e., the converse to (1.44) is false). To see this, consider the following examples:

(1.45) *Examples.* Let $\bar{S}$ be an elliptic curve, so that $\pi_1(\bar{S})\simeq\mathbf{Z}\oplus\mathbf{Z}$. We will construct two elliptic surfaces over $\bar{S}$.

1. Fix a period $\tau_0$ and let $j$ be the constant function $j(\tau_0)$ on $\bar{S}$. Define $\rho\colon\pi_1(\bar{S})\to SL(2,\mathbf{Z})$ by $\rho((1,0))=\rho((0,1))=-I$. Then $\rho$ gives a locally constant sheaf $G$ on $\bar{S}$ which belongs to $j$ (see [8, §8]). Let $\bar{X}$ be the basic member of $\mathscr{F}(j,G)$. Then $\bar{f}\colon\bar{X}\to\bar{S}$ is smooth (so that $\bar{f}$ has trivial local monodromy), and the Serre spectral sequence gives us an exact sequence:

(1.46) $H_2(\bar{X})\to H_2(\bar{S})\to H_1(X_t)_{\pi_1(S)}\to H_1(\bar{X})\to H_2(\bar{S})\to 0.$

Since $H_1(X_t)_{\pi_1(S)}\simeq\mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$ (an easy computation) and $H_2(\bar{X})\to H_2(\bar{S})$ is onto ($\bar{f}$ has a section), we see that $\bar{f}_*\colon H_1(\bar{X})\to H_1(\bar{S})$ is not an isomorphism.

2. Fix the period $\tau_0=(1+i\sqrt{3})/2$ and let $j$ be the constant $j(\tau_0)=0$. Define $\rho\colon\pi_1(S)\to SL(2,\mathbf{Z})$ by:

$$\rho((1,0))=\rho((0,1))=\begin{pmatrix}0&1\\-1&1\end{pmatrix}.$$

This gives a locally constant sheaf $G$ on $\bar{S}$ that belongs to $j$ [8, § 8], and again we take $\bar{X}$ to be the basic member of $\mathscr{F}(j, G)$. $\bar{f}$: $\bar{X} \to \bar{S}$ is smooth and there is no local monodromy, but this time one computes that $H_1(X_t)_{\pi_1(S)} = 0$. Thus, by (1.46), $\bar{f}_*$: $H_1(\bar{X}) \to H_1(\bar{S})$ is an isomorphism.

These examples lead to another partial converse to (1.40):

(1.47) **Proposition.** *Suppose that* $\bar{f}$: $\bar{X} \to \bar{S}$ *has no multiple singular fibers and that* $j \neq 0, 1$. *Then the following are equivalent:*

1. $\bar{f}_*$: $H_1(\bar{X}) \to H_1(\bar{S})$ *is an isomorphism.*
2. $\bar{f}$ *has non-trivial local monodromy.*

*Proof.* $2 \Rightarrow 1$ follows from (1.40). To prove $1 \Rightarrow 2$, assume that $\bar{f}$ is minimal and has trivial local monodromy. Then $\bar{f}$ is smooth because there are no multiple singular fibers, and $j$ is a constant. Thus, in the monodromy representation $\rho$: $\pi_1(\bar{S}) \to SL(2, \mathbf{Z})$, every $\rho(\gamma)$ has a fixed point $\tau_0$ (the period) on $\mathfrak{h}$. Since $j \neq 0, 1$, $\rho(\gamma)$ is $\pm I$, so that $\rho(\gamma) - I$ is either 0 or $-2I$. Then $H_1(X_t)_{\pi_1(\bar{S})} \simeq \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ because the monodromy is non-trivial by (1.44). Since $H_2(\bar{S}) \simeq \mathbf{Z}$, we cannot have a surjection $H_2(\bar{S}) \to H_1(X_t)_{\pi_1(\bar{S})}$. Because $\bar{f}$ is smooth we have the exact sequence (1.46), and this shows that $\bar{f}_*$: $H_1(\bar{X}) \to H_1(\bar{S})$ is not an isomorphism. □

Here is a useful corollary of all the above:

(1.48) **Corollary.** *Let* $\bar{f}$: $\bar{X} \to \bar{S}$ *be an elliptic fibration with non-trivial local monodromy. The following are equivalent:*

1. $H_1(\bar{X})$ *is torsion-free.*
2. *The Néron-Severi group* $NS(\bar{X})$ *is torsion-free.*
3. *All of the integral homology and cohomology groups* $H_i(\bar{X})$ *and* $H^i(\bar{X})$ *are torsion-free.*
4. $\bar{f}_*$: $H_1(\bar{X}) \to H_1(\bar{S})$ *is an isomorphism.*
5. *If* $\bar{f}$ *has* $\geqq 2$ *multiple singular fibers, then their multiplicities are pairwise relative prime.*

*Proof.* $4 \Leftrightarrow 5$ follows from (1.40) and (1.44), and $1 \Leftrightarrow 2$ is well known. $3 \Rightarrow 1$ is trivial and $1 \Rightarrow 3$ is an easy application of Poincaré duality and the universal coefficient theorem. $1 \Rightarrow 4$ follows from (1.41) ($G$ is finite) while $4 \Rightarrow 1$ is true because $H_1(\bar{S})$ is torsion-free. □

Results similar to these have been obtained independently by R. Mandelbaum [12]. Also see [11, 7] and [9] for a deeper look at the topology of elliptic surfaces.

## § 2. Examples

Before we given the examples, we need to recall the arithmetic aspects of our situation. In § 1 we had a minimal elliptic fibration $\bar{f}$: $\bar{X} \to \bar{S}$ which has a section $\sigma_0$ and a non-constant $j$-invariant. The generic fiber of $\bar{f}$, a smooth elliptic curve over $K(\bar{S})$ (the function field of $\bar{S}$), can be defined by a cubic equation in $\mathbf{P}^2_{K(\bar{S})}$:

(2.1) $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3, \quad g_2, g_3 \in K(\bar{S})$.

We will think of this as a point at infinity, (0, 1, 0), together with an affine part defined by the Weierstrass equation:

(2.2) $y^2 = 4x^3 - g_2 x - g_3, \quad g_2, g_3 \in K(\bar{S})$.

Because $\bar{f}$ is minimal, it is the Néron model of (2.2). This means that $\bar{f}\colon \bar{X} \to \bar{S}$ (up to a fiber preserving isomorphism over $\bar{S}$) and the Weierstrass equation (2.2) (up to an isomorphism $(x, y) \to (u^2 x, u^3 y)$, $u \in K(\bar{S})^*$, which transforms (2.2) into the equation:

(2.3) $y^2 = 4x^3 - g_2' x - g_3', \quad g_2' = u^4 g_2, \quad g_3' = u^6 g_3)$

mutually determine each other.

Furthermore, the group of $K(\bar{S})$-rational solutions of (2.2) (with the point at infinity as identity) is naturally isomorphic to the group $\mathfrak{S}$ of sections of $\bar{f}\colon \bar{X} \to \bar{S}$ (with $\sigma_0$ as identity). We will often speak of solutions and Weierstrass equations rather than sections and elliptic fibrations.

*A*. The examples we give are all applications of the following:

(2.4) **Theorem.** *Given the following data:*

1. *A Weierstrass equation* (2.2) *over* $\mathbf{C}(t)$ *with* $p_g = 0$,
2. *Solutions* $\sigma_1, \ldots, \sigma_r$ *of* (2.2),
3. *The order of* $\mathfrak{S}_{\mathrm{tor}}$,

*there is an effective algorithm (described below) to decide whether or not the* $\sigma_i$ *are a basis of* $\mathfrak{S}$ *modulo torsion.*

Working over $\mathbf{C}(t)$ means that $\bar{S} = \mathbf{P}^1$, and the reason for requiring $p_g = 0$ will soon become clear. Before giving the algorithm, let's note that $p_g$ is easy to compute: from (12.6) and (12.7) of [8] we have:

$$\begin{aligned}(2.5)\quad \sigma_0^2 = -(p_g - q + 1) = -(1/12)[&\deg j + 6 \textstyle\sum \nu(I_b^*) \\ &+ 2\nu(\mathrm{II}) + 10\nu(\mathrm{II}^*) + 3\nu(\mathrm{III}) + 9\nu(\mathrm{III}^*) \\ &+ 4\nu(\mathrm{IV}) + 8\nu(\mathrm{IV}^*)]\end{aligned}$$

where $\nu(\mathrm{I}_b^*)$, $\nu(\mathrm{II})$, etc. are the numbers of bad fibers of types $\mathrm{I}_b^*$, II, etc. (and these numbers are easy to determine from $g_2$ and $g_3$ – see [14, III.17]). Since $q = 0$ by (1.48), we can find $p_g$.

Now we give the algorithm. Since $p_g = 0$, $\mathfrak{S}$ has rank $-4 + 2(\#\Sigma) - (\sum_{b>0} \nu(\mathrm{I}_b))$ by (3.23). This must equal the number $r$ of given solutions; otherwise they can't form a basis. When we do have the right number of solutions, then by (1.26), the $\sigma_i$ generate modulo torsion if and only if:

$$\det \langle \sigma_i, \sigma_j \rangle = (\#\mathfrak{S}_{\mathrm{tor}})^2 / \prod_{s \in \Sigma} m_s$$

($m_s$ is defined in (1.22)).

Thus, we need an effective method to compute $\langle \sigma, \sigma' \rangle$ for $\sigma$ and $\sigma'$ in $\mathfrak{S}$. By (1.18), this means computing $(\sigma - \sigma_0) \cdot (\sigma' - \sigma_0)$ (intersection product on $\bar{X}$) and $\sigma \cdot D_s(\sigma')$ for $s \in \Sigma$. To compute the latter, we only have to determine which components of $X_s$ get hit by $\sigma$ and $\sigma'$ (see (1.19)). In $2B$ below we give an effective method for doing this. The computation of $(\sigma - \sigma_0) \cdot (\sigma' - \sigma_0)$ is discussed in $2C$ below.

Using this algorithm to compute the examples in $2E$ is quite straightforward. The only tricky part is determining $\# \mathfrak{S}_{\mathrm{tor}}$, and we discuss this in $2D$ below.

*B*. The problem of determining which component of $X_s$, $s \in \Sigma$, is hit by an element $\sigma \in \mathfrak{S}$ is evidently local on $\bar{S}$. Thus we can assume that we have a solution $\sigma = (\alpha, \beta)$, $\alpha, \beta \in \mathbf{C}((t))$, of a Weierstrass equation:

(2.6) $\quad y^2 = 4x^3 - g_2 x - g_3, \qquad g_2, g_3 \in \mathbf{C}((t)).$

This equation has a local Néron model $\bar{f}$: $\bar{X} \to \mathrm{Spec}(\mathbf{C}[[t]])$. We assume that the special fiber $X_s$ of $\bar{f}$ is not smooth, and we say that the Weierstrass equation (2.6) has type $\mathrm{I}_b$, $\mathrm{I}_b^*$, etc. if $X_s$ has that type.

The solution $\sigma$ gives a section of $\bar{X}$ over $\mathrm{Spec}(C[[t]])$, and we want to know which component of $X_s$ it hits. The difficulty is that constructing $\bar{X}$ from (2.6) is non-trivial (see [14]). However, a first approximation to $\bar{X}$ is fairly easy to obtain. Since we are only interested in (2.6) up to an isomorphism as described in (2.3), we can transform (2.6) into a Weierstrass equation with the following property:

(2.7) *Definition.* A Weierstrass equation (2.6) is called *minimal* if $\mathrm{ord}\, g_2 \geqq 0$, $\mathrm{ord}\, g_3 \geqq 0$ and $\mathrm{ord}\, \Delta = \mathrm{ord}(g_2^3 - 27 g_3^2)$ is as small as possible (i.e., given an isomorphic equation (2.3) with $\mathrm{ord}\, g_2' \geqq 0$, $\mathrm{ord}\, g_3' \geqq 0$, then $\mathrm{ord}\, \Delta' \geqq \mathrm{ord}\, \Delta$).

Take $\bar{X}$ and collapse all of the non-zero components to a point (their intersection matrix is negative definite, so this is possible). This gives a local surface $\bar{Y}$ over $\mathrm{Spec}(\mathbf{C}[[t]])$, and:

(2.8) **Lemma.** *The local surface $\bar{Y}$ is defined by any minimal Weierstrass equation (made projective).*

*Proof.* Néron [14, III.16] shows that $\bar{Y}$ is defined by any standard equation (see [14, III.7]). A minimal equation is standard except in cases $\mathrm{I}_b$ and $\mathrm{I}_b^*$ $(b>0)$, but in (2.18) and (2.24) below, we show that minimal equations of these types are isomorphic, over $\mathrm{Spec}(\mathbf{C}[[t]])$, to standard ones. □

Thus $Y_s$ (the special fiber of $\bar{Y}$) is the cubic $y^2 = 4x^3 - g_2(0)x - g_3(0)$, which has a unique singular point $(a, 0)$ (and $a = 0$ except in case $\mathrm{I}_b$). Our solution $\sigma = (\alpha, \beta)$ gives sections of $\bar{X}$ and $\bar{Y}$ over $\mathrm{Spec}(\mathbf{C}[[t]])$ which are compatible with the collapsing map $\pi$: $\bar{X} \to \bar{Y}$. Determining where $\sigma$ hits $Y_s$ is very easy: just evaluate $(\alpha, \beta)$ at $t = 0$. From this one easily proves:

(2.9.) **Proposition.** *A solution $(\alpha, \beta)$ of a minimal Weierstrass equation hits a non-zero component of $X_s$ if and only if* $\mathrm{ord}(\alpha - a) > 0$. □

We next describe which non-zero component of $X_s$ get hits. We do this case by case, in the following order (based on increasing difficulty): III, III*, IV, IV*,

$I_0^*$, $I_b$ $(b>0)$ and $I_b^*$ $(b>0)$. Types II and II* are omitted because they have only one component of multiplicity one.

When we write an equation like:

$$\alpha = c t^k + \dots$$

we mean that the omitted terms have degree $>k$.

For a minimal equation of type III or III*, we are done by (2.9): for these types, $X_s$ has precisely one non-zero component $C_1$ of multiplicity one (see (1.13)).

Next, we consider minimal Weierstrass equations of types IV and IV*. Using [14, III.17], one easily sees that these equations can be written:

$$\text{(2.10)}\quad \begin{aligned} &\text{IV}: && y^2 = 4x^3 - r t^2 x - s t^2 \\ &\text{IV}^*: && y^2 = 4x^3 - r t^3 - s t^4 \end{aligned}$$

where $r, s \in \mathbf{C}[[t]]$ and $s(0) \neq 0$.

(2.11) **Lemma.** *If* $(\alpha, \beta)$ *is a solution of* (2.10) *with* $\operatorname{ord} \alpha > 0$, *then:*

$$\beta^2 = \begin{cases} -s(0) t^2 + \dots & \text{Type IV} \\ -s(0) t^4 + \dots & \text{Type IV}^*. \end{cases}$$

*Proof.* For type IV*, write $\alpha = u t^k$, where $u$ is a unit and $k \geqq 1$. Then we get:

$$\beta^2 = 4u^3 t^{3k} - r u t^{3+k} - s t^4.$$

If $k=1$, this becomes $\beta^2 = (a \text{ unit}) \cdot t^3$, which is impossible. Thus $k>1$ and $\beta^2$ is as desired. The argument for type IV is similar and even easier. □

(2.12) **Proposition.** *Suppose we have a minimal Weierstrass equation of type IV or IV*, as in* (2.10). *Pick a square root $q$ of $-s(0)$. Then the non-zero components $C_1$ and $C_2$ of multiplicity one in $X_s$ (see* (1.13)) *can be labeled so that a solution* $(\alpha, \beta)$ *of* (2.10) *hits* $C_1$ (resp. $C_2$) *if and only if:*

1. (Type IV) $\beta = q t + \dots$ (resp. $\beta = -q t + \dots$)
2. (Type IV*) $\beta = q t^2 + \dots$ (resp. $\beta = -q t^2 + \dots$).

*Proof.* We will treat type IV – the proof for type IV* is similar. A solution of (2.10) misses the zero component if and only if $\operatorname{ord} \alpha > 0$, and by (2.11), we then have $\beta = \pm q t + \dots$. Thus, we must show that two solutions $(\alpha, \beta)$ and $(\alpha', \beta')$ hit the same non-zero component if and only if $\beta$ and $\beta'$ have the same coefficient of $t$. The crucial fact is that the components of multiplicity one form a group (isomorphic to $\mathbf{Z}/3\mathbf{Z}$), and this group structure is compatible with the addition of sections. Thus $(\alpha, \beta)$ and $(\alpha', \beta')$ hit the same component if and only if $(\alpha, \beta) - (\alpha', \beta')$ hits the zero component. Set $(\alpha_1, \beta_1) = (\alpha, \beta) - (\alpha', \beta')$, and recall that:

$$\text{(2.13)}\quad \alpha_1 = -\alpha - \alpha' + (1/4)[(\beta + \beta')/(\alpha - \alpha')]^2.$$

First, assume that $\beta$ and $\beta'$ have the same coefficient of $t$ (which we can assume to be $q$). Then $(\beta + \beta')/(\alpha - \alpha') = (2qt + \dots)/(\alpha - \alpha')$, so that $\operatorname{ord}(\beta + \beta')/(\alpha - \alpha') \leqq 0$, which implies $\operatorname{ord} \alpha_1 \leqq 0$. Thus, $(\alpha, \beta)$ and $(\alpha', \beta')$ do hit the same component.

If $\beta$ and $\beta'$ have different coefficients of $t$, then $\beta$ and $-\beta'$ have the same coefficients. Since $-(\alpha', \beta')=(\alpha', -\beta')$ the above paragraph shows that $(\alpha, \beta)$ and $-(\alpha', \beta')$ hit the same component, which forces $(\alpha, \beta)$ and $(\alpha', \beta')$ to hit different ones. □

The next case is a minimal Weierstrass equation of type $\mathrm{I}_0^*$. Using [14, III.17] one sees that it can be written:

$$(2.14)\quad y^2=4x^3-rt^2x-st^3$$

where $r$, $s\in\mathbf{C}[[t]]$ and $r(0)^3-27s(0)^2\neq 0$. Then the cubic $4x^3-r(0)x-s(0)$ has three distinct roots which we call $r_1$, $r_2$ and $r_3$.

(2.15) **Proposition.** *Suppose we have a minimal Weierstrass equation of type* $\mathrm{I}_0^*$ *as in* (2.14). *Then the non-zero components* $C_1$, $C_2$ *and* $C_3$ *of multiplicity one in* $X_s$ *(see* (1.13)) *can be labeled so that a solution* $(\alpha, \beta)$ *of* (2.14) *hits* $C_i$ *if and only if* $\alpha=r_i t+\dots$.

*Proof.* Let $(\alpha, \beta)$ be a solution of (2.14) that misses the zero component. Using (2.9) to write $\alpha=ut+\dots$, we get $\beta^2=(4u^3-r(0)u-s(0))t^3+\dots$, so that $u\in\{r_1, r_2, r_3\}$ and $\operatorname{ord}\beta\geqq 2$. Then one proceeds as in the proof of (2.12), taking the difference of two solutions and using (2.13). □

We move on to the case of a minimal Weierstrass equation of type $\mathrm{I}_b$, $b>0$, which can be written:

$$(2.16)\quad y^2=4x^3-g_2x-g_3$$

where $g_2$, $g_3\in\mathbf{C}[[t]]^*$ and $\operatorname{ord}\Delta=b$ (where $\Delta=g_2^3-27g_3^2$). $Y_s$ is defined by $y^2=4x^3-g_2(0)x-g_3(0)$ and has a singular point $(a, 0)$, where $a=-3g_3(0)/2g_2(0)$.

(2.17) **Lemma.** *Let* $(\alpha, \beta)$ *be a solution of* (2.16) *that misses the zero component. Then* $\operatorname{ord}(12\alpha^2-g_2)>0$. *If we write* $12\alpha^2-g_2=ct^k+\dots$, $c\neq 0$, *then:*

1. *If $b$ is odd, then $2k<b$ and $\beta^2=(c^2/48a)t^{2k}+\dots$*
2. *If $b$ is even and $2k<b$, then $\beta^2=(c^2/48a)t^{2k}+\dots$.*

*Proof.* From (2.9) we know that $\alpha(0)=a$. Since $g_2(0)=12a^2$, we see that $\operatorname{ord}(12\alpha^2-g_2)>0$. Then write $g_2=12v^2$ where $v=a+\dots$, so that $\Delta=g_2^3-27g_3^2=27(8v^3+g_3)(8v^3-g_3)$. If $\Delta=mt^b+\dots$, then $8v^3+g_3=(m/27\cdot 16a^3)t^b+\dots$ (since $g_3(0)=-8a^3$). Also $12\alpha^2-g_2=12(\alpha+v)(\alpha-v)$, so that $\alpha-v=(c/24a)t^k+\dots$.

Let $f(x)=4x^3-g_2x-g_3$. Then we compute that $f(v)=-(8v^3+g_3)$ and $f'(v)=0$, so that:

$$\begin{aligned}(2.18)\quad \beta^2&=f(\alpha)=f(v)+f'(v)((c/24a)t^k+\dots)\\&\quad+(1/2)f''(v)((c/24a)t^k+\dots)^2+\dots\\&=-(m/27\cdot 16a^3)t^b+\dots+(c^2/48a)t^{2k}+\dots\end{aligned}$$

and the lemma follows easily. □

(2.19) **Proposition.** *Suppose we have a minimal Weierstrass equation of type* $\mathrm{I}_b$, $b>0$, *as in* (2.16), *and pick a square root* $q$ *of* $3a$. *Then the non-zero components* $C_1, \dots, C_{b-1}$ *of* $X_s$ *(see* (1.13)) *can be labeled so that if a solution* $(\alpha, \beta)$ *of* (2.16) *misses the zero component, then:*

1. $(\alpha, \beta)$ *hits* $C_{b/2}$ *if and only if* $2\operatorname{ord}\beta \geqq b$.
2. $(\alpha, \beta)$ *hits* $C_k$ *or* $C_{b-k}$ *if and only if* $\operatorname{ord}\beta = k$, $2k < b$.

*In this case, we can write* $12\alpha^2 - g_2 = c t^k + \ldots$, $c \neq 0$, *and then* $(\alpha, \beta)$ *hits* $C_k$ (resp. $C_{b-k}$) *if and only if* $\beta = (c/4q) t^k + \ldots$ (resp. $\beta = -(c/4q) t^k + \ldots$).

*Proof.* This proof will make extensive use of [14], including notation. Let $v$ be as in (2.17). Then the change of coordinates $(x, y, z) \to (x - vz, y, z)$ transforms (2.16) (made projective) into the equation:

$$(2.20) \quad A\colon\ y^2 z = 4x^3 + 12 v x^2 z - (8v^3 + g_3) z^3.$$

Let $N = 12v$, $M = -(8v^3 + g_3)$. Then the Néron model $\overline{X}$ is built from the equations

$$A_i\colon\ y^2 z = 4t^i x^3 + N x^2 z + M t^{-2i} z^3$$

for $1 \leqq i \leqq \ell$, where $b = 2\ell$ or $2\ell + 1$ (see [14, §§9–10]). If $A_i^0$ is the special fiber of $A_i$, then $A_i^0$ is defined by $z(y^2 - 12ax^2) = 0$ for $2i < b$ and $A_{b/2}^0$ is defined by $z(y^2 - 12ax^2 + (m/27 \cdot 16a^3) z^2) = 0$.

We have projection maps $\pi_i\colon X_s \to A_i^0$, and we label the components $C_1, \ldots, C_{b-1}$ so that $\pi_{b/2}$ takes $C_{b/2}$ onto the conic:

$$(2.21) \quad y^2 - 12ax^2 + (m/27 \cdot 16a^3) z^2 = 0$$

and $\pi_i$ takes $C_i$ (resp. $C_{b-i}$) onto the line $y = 2qx$ (resp. $y = -2qx$). See [14, §9] and the table of generic points [14, p. 104].

The solution $(\alpha, \beta)$ of (2.16) gives a solution $(\alpha - v, \beta, 1)$ of (2.20) and hence a solution $\sigma_i = (\alpha - v, \beta, t^i)$ of $A_i$. We have to figure out where $\sigma_i(0)$ lands on $A_i^0$.

If $\operatorname{ord}\beta = k$, $2k < \beta$, by (2.17) we can write $12\alpha^2 - g_2 = c t^k + \ldots$, $c \neq 0$, and then the proof of (2.17) shows that

$$\begin{aligned}\sigma_k &= ((c/24a) t^k + \ldots, \pm (c/4q) t^k + \ldots, t^k)\\ &= ((c/24a) + \ldots, \pm (c/4q) +, 1).\end{aligned}$$

When $t = 0$, this is on either $y = 2qx$ or $y = -2qx$ in $A_k^0$, depending on the sign of $\beta$.

When $2\operatorname{ord}\beta = k$, the argument is similar: one uses (2.17) and (2.18) to verify that $\sigma_{b/2}$ hits $A_{b/2}^0$ on the conic (2.21). The final case, also treated similarly, is when $2\operatorname{ord}\beta > k$. $\square$

Finally, we have the case of a minimal Weierstrass equation of type $\mathrm{I}_b^*$, $b > 0$, which can be written:

$$(2.22) \quad y^2 = 4x^3 - r t^2 x - s t^3$$

where $r, s \in \mathbf{C}[[t]]^*$ and $\operatorname{ord}(r^3 - 27s^2) = b$. Let $r^3 - 27s^2 = m t^b + \ldots$. The cubic $4x^3 - r(0)x - s(0)$ has a double root $a = -3s(0)/2r(0)$ (and the other root is $-2a$).

(2.23) **Lemma.** *Let $(\alpha, \beta)$ be a solution of (2.22) which misses the zero component of $X_s$. Then either $\alpha = -2at + \dots$ or $\alpha = at + \dots$, and if $\alpha = at + \dots$, then:*

1. *When $b$ is odd, $\beta^2 = (-m/27 \cdot 16a^3)\, t^{b+3} + \dots$*
2. *When $b$ is even,*

$$12\alpha^2 - g_2 = \pm(\sqrt{m}/3a)\, t^{(b+4)/2} + \dots .$$

*Proof.* We know that $\operatorname{ord} \alpha > 0$, and then $\alpha = -2at + \dots$ or $\alpha = at + \dots$ follows as in (2.15).

Since $r(0) = 12a^2$, we can write $r = 12v^2$ where $v = a + \dots$. Set $\alpha - vt = ct^k + \dots$, where $c \neq 0$ and $k > 1$. Then manipulating as in the proof of (2.17), we get:

$$(2.24) \quad \beta^2 = (-m/27 \cdot 16a^3)\, t^{b+3} + \dots + 12ac^2 t^{2k+1} + \dots .$$

From this, the lemma follows easily. □

(2.25) **Proposition.** *Suppose we have a minimal Weierstrass equation of type $\mathrm{I}_b^*$, $b > 0$, as in (2.22). Let $q$ be a square root of $-m/3a$ ($b$ odd) or of $m$ ($b$ even). Then the non-zero components $C_1$, $C_{b+3}$, $C_{b+4}$ of multiplicity one in $X_s$ (see (1.13)) can be labeled so that if $(\alpha, \beta)$ is a solution of (2.22) missing the zero component, then:*

1. *$(\alpha, \beta)$ hits $C_1$ if and only if $\alpha = -2at + \dots$.*
2. *$(\alpha, \beta)$ hits $C_{b+3}$ if and only if $\alpha = at + \dots$ and $\beta = (q/12a)\, t^{(b+3)/2} + \dots$ ($b$ odd) or $12\alpha^2 - g_2 = (q/3a)\, t^{(b+4)/2} + \dots$ ($b$ even).*
3. *$(\alpha, \beta)$ hits $C_{b+4}$ if and only if $\alpha = at + \dots$ and $\beta = -(q/12a)\, t^{(b+3)/2} + \dots$ ($b$ odd) or $12\alpha^2 - g_2 = -(q/3a)\, t^{(b+4)/2} + \dots$ ($b$ even).*

*Proof.* Let $v$ be as in (2.23). Then $(x, y, z) \to (x - vtz, y, z)$ transforms (2.22) into:

$$(2.26) \quad A\colon\ y^2 z = 4x^3 - 12vtx^2 z - (8v^3 t^3 + g_3) z^3.$$

Then $\bar{X}$ is built out of equations $A_1, \dots, A_{b+2}$, each of which is a transform of $A$ (see [14, III.12]). The cases $b$ even and $b$ odd are treated separately. In each case, the proof of this proposition is similar to the proof of (2.19).

The component $C_1$ maps onto a component of $A_2^0$ under $\pi_2\colon \bar{X} \to A_2$; similarly $C_{b+3}$ and $C_{b+4}$ correspond to components of $A_{b+2}^0$ under $\pi_{b+2}$ (see [13, III.12]). $A$ solution $\sigma$ of (2.22) gives solution $\sigma_2$ of $A_2$ and $\sigma_{b+2}$ of $A_{b+2}$. Using (2.23) and (2.24), it is easy to see which components these hit. □

*Remark.* In the proofs of (2.12) and (2.15) (types IV, IV* and $\mathrm{I}_0^*$), we used results from Néron [14] on the number of components of multiplicity one and their group structure. Our methods can be used to give elementary (though cumbersome) proofs of these results and results for some other types (for example, the group structure of the four components of multiplicity one in a fiber of type $\mathrm{I}_b^*$). One the other hand, one can use Néron to prove (2.12) and (2.15) (see (2.19) and (2.25)).

It is clear that all of the above results are valid over any algebraically closed field of characteristic different from 2 or 3.

*C*. Given $\sigma$ and $\sigma'$ in $\mathfrak{S}$, we need to compute the intersection number $\sigma \cdot \sigma'$, where we regard $\sigma$ and $\sigma'$ as curves on $\bar{X}$. If $\sigma = \sigma'$, then $\sigma \cdot \sigma' = \sigma'^2 = \sigma_0^2$, and this is easy to compute by (2.5). If $\sigma \neq \sigma'$, then $\sigma \cdot \sigma'$ is a sum of local contributions from the points $P$ of $\sigma \cap \sigma'$ on $\bar{X}$.

The difficulty is that, in practice, we don't have $\bar{X}$; rather, we have a Weierstrass equation (2.2) and two explicit solutions $\sigma$ and $\sigma'$. But one can still compute $\sigma \cdot \sigma'$ from this data.

If $\sigma$ and $\sigma'$ meet at a point $P$ which lies either on a good fiber or the zero component of a bad fiber, then near $P$, $\bar{X}$ is defined by any minimal Weierstrass equation. Then one can use this equation to compute the local contribution at $P$ to $\sigma \cdot \sigma'$.

If $\sigma$ and $\sigma'$ meet at a point $P$ on a non-zero component of $X_s$, $s \in \Sigma$, there are two things one can do. First, one could use Neron [14] and push $\sigma$ and $\sigma'$ down to the appropriate $A_i$ and compute these (see the proofs of (2.19) and (2.25) for examples of "pushing down"). The other way is to note that $\sigma - \sigma'$ (the difference in $\mathfrak{S}$) meets $\sigma_0$ at infinity on the zero-component of $X_s$, and the local contribution of $(\sigma - \sigma') \cdot \sigma_0$ at infinity is the same as the local contribution of $\sigma \cdot \sigma'$ at $P$ (subtracting $\sigma'$ fiberwise gives a birational map of $\bar{X}$ to itself defined in a neighborhood of $P$, so it preserves local intersection multiplicities).

*D*. We do not know a systematic method for determining torsion. But there are several tactics which work well. The first is the fact that for $s \in \Sigma$ we have (by [16, 19]) an injection:

(2.27) $\quad \mathfrak{S}_{\mathrm{tor}} \to (X'_s)_{\mathrm{tor}}$.

If $X_s$ is not of type $\mathrm{I}_b$, then $X'_s$ is an extension of $\mathbf{C}$ by the finite group $G_s$ of components of multiplicity one. So in this case we have an injection:

(2.28) $\quad \mathfrak{S}_{\mathrm{tor}} \to G_s$.

For example, consider the equation $y^2 = 4x^3 - 3t^3 x + t^4$ over $\mathbf{C}(t)$. The bad fibers occur at $t = 0, 1$ and $\infty$ and are of types IV*, $\mathrm{I}_1$ and III respectively. The fiber over $t = 0$ (resp. $t = \infty$) tells us, via (2.28), that any torsion has order 3 (resp. 2). Thus $\mathfrak{S}$ has no torsion.

If all the bad fibers are of type $\mathrm{I}_b$, then (2.27) doesn't give much information. But $\mathfrak{S}_0$ is torsion-free (see (1.17)), so that we still have an injection:

$$\mathfrak{S}_{\mathrm{tor}} \to \bigoplus_{s \in \Sigma} G_s.$$

so all torsion is killed by the l.c.m. of the exponents of the $G_s$.

Finally, the bilinear form $\langle \ , \ \rangle$ of §1 can be used to give more detailed information. For example, consider the equation $y^2 = x(x-1)(x - t^2 - c)$ over $\mathbf{C}(t)$, where $c \in \mathbf{C} - \{0, 1\}$. The bad fibers occur at $t = \pm\sqrt{-c}$, $\pm\sqrt{1-c}$ and $\infty$ and are of type $\mathrm{I}_2$ except for $t = \infty$ which is of type $\mathrm{I}_4$. Let $\sigma$ be a torsion solution, and assume $2\sigma \neq \sigma_0$. Then $2\sigma$ hits the zero component except at $t = \infty$ where it must hit $C_2$. Thus $\langle 2\sigma, 2\sigma \rangle = -(2\sigma - \sigma_0)^2 - 1$ (using (1.25) and (1.26)), so $\langle 2\sigma, 2\sigma \rangle = 1 + 2(2\sigma \cdot \sigma_0)$ since $\sigma^2 = \sigma_0^2 = -1$. Since $2\sigma$ is torsion, $\langle 2\sigma, 2\sigma \rangle = 0$ and we get a contradiction. Thus, the only torsion solutions are the obvious 2-torsion ones: $(0,0)$, $(1,0)$, $(t^2 + c, 0)$.

*E*. Now that the algorithm is complete we give the examples. We will do the first two in detail and then just list the others.

In all these examples one has $q=0$ and, via (2.5), $p_g=0$ and $\sigma_0^2=-1$.

*Example 1*. Consider the equation over $\mathbf{C}(t)$:

(2.29) $y^2=4x^3-3t(t-B)^2x-t(t-B)^3$

where $B\in\mathbf{C}-\{0,1\}$. Here $\Delta=27t^2(t-1)(t-B)^6$ and $j=t/(t-1)$. The bad fibers occur at $t=0,1,B$ and $\infty$, and are respectively of types II, $\mathrm{I}_1$, $\mathrm{I}_0^*$ and III. $\mathfrak{S}$ has rank 3 and no torsion by (2.28) (use the fiber of type II). The algorithm (2.4) shows that a basis of $\mathfrak{S}$ will have determinant (with respect to $\langle\ ,\ \rangle$) equal to 1/8.

Let $r_1$, $r_2$, and $r_3$ be the distinct roots of $4x^3-3Bx-B$, and let $s_i$ be a square root of $-4r_i^3/B$. Then $\sigma_i=(r_i(t-B), s_i(t-B)^2)$ is a solution of (2.29). We claim that these form a basis of $\mathfrak{S}$.

Over $t=0$ and $t=1$ it is clear that $\sigma_0$, $\sigma_1$, $\sigma_2$ and $\sigma_3$ hit the fiber in different places.

At $t=B$, we use $T=t-B$ as a local parameter, so (2.29) becomes $y^2=4x^3-3T^2(B+T)x-T^3(B+T)$ (which is minimal) and $\sigma_i$ is $(r_iT, s_iT^2)$. Then (2.15) shows that each $\sigma_i$ hits a different component over $t=B$.

At $t=\infty$, we transform the equation to $y^2=4x^3-3t(t-B)^{-2}x-t(t-B)^{-3}$, and using $T=(t-B)^{-1}$ as a local parameter, this becomes the minimal equation $y^2=4x^3-3T(1+BT)x-T^2(1+BT)$, with solutions $\sigma_i=(r_iT, s_iT)$. By (2.9), all the $\sigma_i$ hit $C_1$. To see if they meet there, set $(\alpha,\beta)=\sigma_i-\sigma_j$. Then $\alpha=-(r_i-r_j)T+1/4(s_i+s_j)^2(r_i-r_j)^{-2}$, which does not have a pole at $T=0$. Thus, the $\sigma_i$ do not meet over $\infty$.

It is also clear that $\sigma_0$, $\sigma_1$, $\sigma_2$, $\sigma_3$ cannot meet anywhere else, so that $\sigma_i\cdot\sigma_j=0$ for $i\neq j$, $0\leqq i,j\leqq 3$. Then from (1.18) and (1.19) we see that the matrix $\langle\sigma_i,\sigma_j\rangle$ is:

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

which has determinant 1/8, as desired.

*Example 2*. Consider the equation over $\mathbf{C}(t)$:

(2.30) $y^2=x(x-1)(x-t^2-c)$

where $c\in\mathbf{C}-\{0,1\}$. Since $\Delta=16\lambda^2(\lambda-1)^2$ (where $\lambda=t^2+c$) the bad fibers are at $t=\pm\sqrt{-c}$, $\pm\sqrt{1-c}$ and $\infty$, all of type $\mathrm{I}_2$ except for $t=\infty$, which is of type $\mathrm{I}_4$. The rank is 1, and in 2*D* we determined the torsion. Thus $\sigma$ is a basis modulo torsion if and only if $\langle\sigma,\sigma\rangle=1/4$.

Let $m=\sqrt{1-c}-\sqrt{-c}$. Then $\sigma=(m(t+\sqrt{-c}), im(t+\sqrt{-c})(t-\sqrt{1-c}))$ is a solution of (2.30). We need to show that $\langle\sigma,\sigma\rangle=1/4$.

The coordinate change $(x,y)\to(x-(\lambda+1)/3, 2y)$ transforms (2.28) to the Weierstrass equation:

(2.31) $$y^2=4x^3-(4(\lambda^2-\lambda+1)/3)x -4(\lambda+1)(\lambda-2)(2\lambda-1)/27$$

where $\lambda = t^2 + c$. This is minimal at $t = \pm\sqrt{-c}$, $\pm\sqrt{1-c}$. Our solution becomes $\sigma = (m(t+\sqrt{-c}) - (\lambda+1)/3, 2im(t+\sqrt{-c})(t-\sqrt{1-c}))$.

At $t = \pm\sqrt{-c}$, $Y_s$ has singular point $(-1/3, 0)$. Thus, at $t = \sqrt{-c}$, $\sigma$ hits $C_0$, while at $t = -\sqrt{-c}$ $\sigma$ hits $C_1$ (using (2.9)). At $t = \pm\sqrt{1-c}$, $Y_s$ has a singular point $(1/3, 0)$. Since $m(\pm\sqrt{1-c} + \sqrt{-c}) = 1$ or $-m^2$, $\sigma$ hits $C_1$ at $t = \sqrt{1-c}$ while at $t = -\sqrt{1-c}$, it hits $C_0$ (again, this is (2.9)).

At $t = \infty$, we use $T = 1/t$ as local parameter, and (2.31), after multiplying $g_2$ (resp. $g_3$) by $T^4$ (resp. $T^6$), becomes $y^2 = 4x^3 - ((4/3) + \dots)x - ((8/27) + \dots)$. $Y_s$ has a singular point $(-1/3, 0)$. $\sigma$ becomes $(-(1/3) + mT + \dots, 2imT + \dots)$, so that by (2.19) $\sigma$ hits $C_1$.

It is clear that $\sigma \cdot \sigma_0 = 0$, and then we easily get $\langle \sigma, \sigma \rangle = 1/4$. Thus $\sigma$ generates modulo torsion, and we know what the torsion is.

The table below gives six more examples. In each of theses cases $\mathfrak{S}_{\mathrm{tor}} = 0$, so we actually give a basis for $\mathfrak{S}$.

(2.32) Solutions of Weierstrass equations over $\mathbf{C}(t)$

| *Equation* | *Basis of* $\mathfrak{S}$ |
|---|---|
| 1. $y^2 = 4x^3 - 3t^3x + t^4$ | $(0, t^2)$ |
| 2. $y^2 = 4x^3 - 3tx + t$ | $(1/3, \sqrt{4/27})$ |
| 3. $y^2 = 4x^3 - 3tx + 1$ | $(0, 1)$ |
| 4. $y^2 = 4x^3 - 4t^2x + t^2$ | $(0, t), (t, t)$ |
| 5. $y^2 = 4x^3 - 3t(t-1)^2x$ | $(t-1, 2i(t-1)^2)$, |
| $\quad - t(t-1)^3$ | $(-(1/2)(t-1), (1/\sqrt{2})(t-1)^2)$ |
| 6. $y^2 = 4x^3 - 4t^2x + 4$ | $(0, 2), (t, 2), (-1, 2t)$, |
| | $(\lambda, 2\lambda^2 t), \lambda = e^{\pi i/3}$. |

Determining that the torsion is zero in Eqs. 3 and 6 is similar to what was done for Example 2 above; in the other cases it is a simple consequence of (2.28).

Another example is a generic elliptic surface with $p_g = 0$. Here $\bar{X}$ is $\mathbf{P}^2$ blown up at nine points on a cubic, and there are 12 singular fibers of type $\mathrm{I}_1$. The 9 exceptional curves are sections $\sigma_i$, $0 \leqq i \leqq 8$ (where $\sigma_0$ is the zero section). Then one computes that $\det \langle \sigma_i, \sigma_j \rangle_{1 \leqq i,j \leqq 8} = 9$, so that the $\sigma_i$ generate a subgroup of index 3 in $\mathfrak{S}$.

*Remark.* The solutions of Examples 1 and 2 and Eqs. 1, 2 and 5 of (2.32) were found by W. Hoyt (who conjectured that they were bases). In each of these cases, C. Schwartz [17] showed that the solutions are independent in $\mathfrak{S}$, and that the solution $\sigma$ of Example 2 does generate modulo torsion [18]. His methods require involved calculations with Picard-Fuchs equations and automorphic forms.

## §3. Parabolic Cohomology, Automorphic Forms and Hodge Theory

*A.* As in §1, we let $f: X \to S$ be the smooth part of an elliptic fibration, with non-constant $j$-invariant and zero-section $\sigma_0$. The universal covering space of $S$ may be identified with the upper half-plane $\mathfrak{h}$. Pulling back $X$ to $\mathfrak{h}$, one obtains a diagram

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \mathfrak{h} \arrow[r, "\pi"] & S \end{tikzcd}$$

There is a simple description of $\tilde{X}$ in terms of periods. Let $\mathscr{F}^1 = f_* \Omega^1_{X/S}$ (the first Hodge filtration bundle over $S$) and let $\omega_0$ be any generating section of $\tilde{\mathscr{F}}^1 = \pi^* \mathscr{F}^1 = g_* \Omega^1_{\tilde{X}/\mathfrak{h}}$. Choose a basis $\{u_1, u_2\}$ for the constant sheaf $R^1 g_* \mathbf{Z}$ with cup-product $(u_1, u_2) = -1$. Then $\omega = (u_1, \omega_0)^{-1} \omega_0$ gives value 1 when paired with $u_1$, and $\tau = -(u_2, \omega)$ is a holomorphic function on $\mathfrak{h}$ with values in $\mathfrak{h}$. For any $z \in \mathfrak{h}$, the numbers 1 and $\tau(z)$ generate the period lattice for $X_{\pi(z)}$, and one obtains the formula $\omega = \tau u_1 + u_2$ as a section of $\tilde{\mathscr{V}} = \mathcal{O}_{\mathfrak{h}}(R^1 g_* \mathbf{C})$. This also allows one to represent the universal cover of $\tilde{X}$ as $\mathfrak{h} \times \mathbf{C}$ in a natural way.

(3.1) *Remark.* A simple example is the case where $X$ is given by the Legendre equation $y^2 = x(x-1)(x-t)$ $(S = \mathbf{P}^1 - \{0, 1, \infty\})$. Here it can be arranged that $\tau(z) = z$.

The elements $u_1$ and $u_2$ may be construed as generators for $\tilde{V} = H^1(\tilde{X}, \mathbf{C})$ under the natural isomorphism

$$\tilde{V} \xrightarrow{\sim} H^0(\mathfrak{h}, R^1 g_* \mathbf{C}).$$

With respect to this basis, local sections of $\tilde{\mathscr{V}}$ will be represented by column vector valued functions. We will also identify sections of $\mathcal{O}_S(V)$, where $V = R^1 f_* \mathbf{C}$, with their pullbacks to $\mathfrak{h}$.

The fundamental group $\Gamma$ of $S$ acts as deck transformations on $\mathfrak{h}$, hence on $\tilde{X}$, inducing the *monodromy representation:*

$$M\colon \Gamma \to \mathrm{Aut}_{\mathbf{Z}}(\tilde{V}) = SL(2, \mathbf{Z})$$
$$M(\gamma) = (\gamma^{-1})^*.$$

The locally constant sheaf $V$ (also $V_{\mathbf{Z}}$, $V_{\mathbf{Q}}$, etc.) may be reconstituted from the monodromy by identifying $(z, v)$ with $(\gamma z, M(\gamma) v)$ in $\mathfrak{h} \times \tilde{V}$.

There is the usual notion of parabolic subgroups of $\Gamma$, of which the conjugacy classes are in one-to-one correspondence with the points of $\Sigma$. We can identify the sheaf cohomology group used in §1 with parabolic group cohomology (as defined in [21]):

(3.2) **Lemma.** *Let $V$ be any locally constant sheaf on the non-singular algebraic curve $S$, with fundamental group $\Gamma \subset SL(2, \mathbf{R})$, $\tilde{V}$ the associated $\Gamma$-module, and*

$$j\colon S \to \bar{S}$$

*the inclusion of $S$ in its smooth completion. Then there is a commutative diagram with exact horizontal rows:*

$$\begin{tikzcd} 0 \arrow[r] & H^1_P(\Gamma, \tilde{V}) \arrow[r] \arrow[d, dashed, "\simeq"] & H^1(\Gamma, \tilde{V}) \arrow[r] \arrow[d, "\simeq"] & \bigoplus_{\Gamma_0} H^1(\Gamma_0, \tilde{V}) \arrow[d, "\simeq"] \\ 0 \arrow[r] & H^1(\bar{S}, j_* V) \arrow[r] & H^1(S, V) \arrow[r] & \bigoplus_{s \in \Sigma} H^1(\Delta^*(s), V) \end{tikzcd}$$

*where $\Gamma_0$ ranges over conjugacy class representatives of parabolic subgroups of $\Gamma$, the top row defines the parabolic cohomology group $H^1_P(\Gamma, \tilde{V})$, and $\Delta^*(s)$ is a small punctured disc about the point s.*

*Proof.* (Cf. [25, §12].) $\square$

*B.* In §1, we defined $\delta(\sigma)$ for $\sigma \in \mathfrak{S}$, and the pairing $\langle \sigma, \sigma' \rangle$. We will now show that these coincide respectively with the cohomology classes of automorphic forms naturally associated to the sections, and the generalized Eichler bilinear form.

We begin by mentioning the way in which the cohomology groups in Lemma (3.2) are computed using differential forms, leaving the full discussion for Part C. On $S$, we have the holomorphic deRham complex:

$$\Omega^{\bullet}_S(V)\colon\ \mathcal{O}_S(V) \xrightarrow{d} \Omega^1_S(V)$$

forming a resolution of $V$. This can be extended to a resolution of $j_* V$ by taking:

$$(3.3)\qquad \bar{\mathscr{V}} \to d\bar{\mathscr{V}},$$

where $\bar{\mathscr{V}}$ denotes the *canonical extension* [2, p. 95] of $\mathscr{V} = \mathcal{O}_S(V)$. Let $\bar{\mathscr{V}}(*D)$ denote the sheaf of germs of meromorphic sections of $\bar{\mathscr{V}}$ having arbitrary poles only on the support of the fixed effective divisor $D$. Then:

$$(3.4)\qquad \bar{\mathscr{V}}(*D) \to d[\bar{\mathscr{V}}(*D)]$$

will be called the complex of *forms of the second kind* with values in $\bar{\mathscr{V}}$ and poles on $D$. Since the complex (3.4) evidently resolves $j_* V$, we have:

(3.5) **Lemma.** *The inclusion of the complex* (3.3) *in* (3.4) *induces an isomorphism on hypercohomology.* $\square$

Thus, we should regard differentials of the second kind as intrinsically representing cohomology classes on $\bar{S}$ (as opposed to $S$). However, the presence of poles will cause a technical problem in computing cup products.

Let $\sigma$ be a section of $X$. Then $\sigma$ determines a section $\tilde{\sigma}\colon \mathfrak{h} \to \tilde{X}$, which may be lifted to a mapping:

$$(1, F)\colon\ \mathfrak{h} \to \mathfrak{h} \times \mathbf{C},$$

well-defined up to periods. $F$ may be considered to be an explicit expression for the *normal function* associated to $\sigma - \sigma_0$, regarding $X$ as its own system of Jacobian varieties. One should regard $F$ as giving the image of the vector $F(z)\, u_1$ under the mapping:

$$\psi\colon\ \tilde{\mathscr{V}} \to (\tilde{\mathscr{F}}^1)^*$$

$$\psi \begin{bmatrix} x \\ y \end{bmatrix} \omega = x - y\tau$$

induced by cup-product. If $\gamma \in \Gamma$, and:

$$M(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

set $j(M(\gamma),\tau)=c\tau+d$. Then it is easy to show that $F$ satisfies the functional equation:

(3.6) $j(M(\gamma^{-1}),\tau)\,F(\gamma^{-1}z)=F(z)+q_\gamma\tau+r_\gamma$

for suitable integers $q_\gamma$, $r_\gamma$. Defining the periods for $F$ as:

$$\beta(\gamma)=\begin{bmatrix} r_\gamma \\ -q_\gamma \end{bmatrix}\in\tilde{V},$$

the vector whose image in $(\tilde{\mathscr{F}}^1)^*$ is $q_\gamma\tau+r_\gamma$, one sees immediately using (3.6), that $\beta(\gamma)$ is a 1-cocycle for $\Gamma$ acting on $\tilde{V}$, so it determines an element of $H^1(\Gamma,\tilde{V})$ (which will soon be seen to be parabolic).

One associates to every section a generalized automorphic form in the sense of Hoyt [4] according to the following recipe. For any function $F$ on $\mathfrak{h}$, we may define its derivative with respect to $\tau$:

$$\frac{dF}{d\tau}=F'(z)\,\tau'(z)^{-1}.$$

Note that $d/d\tau$ is a meromorphic differential operator with poles at the ramification points of $\tau$.

(3.7) **Proposition** [4]. *If $F$ satisfies Eq. (3.6), then $G=d^2F/d\tau^2$ is a meromorphic automorphic form of weight* 3 *with respect to* $(\tau, M)$, *i.e.,*

$$G(\gamma z)=j(M(\gamma),\tau)^3\,G(z)\qquad\forall\gamma\in\Gamma.\quad\square$$

The space of all meromorphic functions satisfying the above transformation rule will be denoted $A_3(M,\tau)$. Similarly, as:

$$\varphi_G=G(z)\begin{bmatrix}\tau\\1\end{bmatrix}d\tau=d\left(dF/d\tau\begin{bmatrix}\tau\\1\end{bmatrix}-F\begin{bmatrix}1\\0\end{bmatrix}\right)$$

is invariant under $\Gamma$, it descends to $S$ as a meromorphic 1-form of the second kind with values in $V$; in fact, (see (3.24)):

$$\varphi_G\in\Gamma(\bar{S},d[\bar{\mathscr{V}}(*D)]),$$

where $D$ is the ramification divisor on $\bar{S}$ for $\tau$. Implicit in this last statement is the assertion that there is a proper notion of ramification at the points of $\Sigma$, a matter which will be settled in (3.17). Note also that $\operatorname{ord}_z d\tau=\operatorname{ord}_{\gamma z} d\tau$ for all $\gamma\in\Gamma$, so for $s\in S$, $\operatorname{ord}_s d\tau$ can be defined.

The remainder of § 3B will be devoted to the relation among $F$, $\varphi_G$, and $\delta(\sigma)$ [defined in (1.6)]. Let $G\in A_3(M,\tau)$, with $\varphi_G$ of the second kind, and fix a base point $z_0\in\mathfrak{h}$. The $\tilde{V}$-valued meromorphic function:

(3.8) $\phi(z)=\int_{z_0}^{z}\varphi_G$

is well-defined, independent of choice of path from $z_0$ to $z$. Its *period cocycle*, given by:

$$\alpha(\gamma)=M(\gamma)\,\phi(z)-\phi(\gamma z)\in C^1(\Gamma,\tilde{V})$$

represents the deRham cohomology class of $\varphi_G$ in $H^1(\Gamma, \tilde{V}) \simeq H^1(S, V)$, which we write as $\alpha = [\varphi_G]$. Of course, Lemma (3.5) implies that $\varphi_G$ naturally represents an element of the parabolic cohomology $H^1(\bar{S}, j_* V)$. If $G$ is a cusp form, this observation is visible at the group cocycle level.

The periods for $F$ and $\varphi_G$ are related by:

(3.9) **Proposition.** *$\beta = \alpha$ in $H^1(S, V)$.*

*Proof.* It can be seen directly that we may choose $F$ and $\phi$ so that:

$$\begin{aligned} F(z) &= {}^t\begin{bmatrix} \tau \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \phi(z) \\ &= (\omega \cup \phi)(z) \qquad \text{(cup-product in } \tilde{V}). \end{aligned}$$

Putting $\Theta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we have from (3.6):

$$\begin{aligned} q_\gamma \tau + r_\gamma &= {}^t\omega(z)\,\Theta\,\phi(z) - j(M\gamma^{-1}, \tau)\,{}^t\omega(\gamma^{-1} z)\,\Theta\,\phi(\gamma^{-1} z) \\ &= {}^t\omega(z)\,\Theta\,\phi(z) - j(M\gamma^{-1}, \tau)[j(M\gamma^{-1}, \tau)^{-1}\,{}^t\omega(z)\,{}^tM\gamma^{-1}] \\ &\quad \cdot \Theta[M\gamma^{-1}\phi(z) - \alpha(\gamma^{-1})] \\ &= {}^t\omega(z)\,{}^tM\gamma^{-1}\,\Theta\,\alpha(\gamma^{-1}) = -{}^t\begin{bmatrix} \tau \\ 1 \end{bmatrix} \Theta\,\alpha(\gamma) \end{aligned}$$

since ${}^tM\Theta M = \Theta$ (flatness of cup-product) and $M\gamma\alpha(\gamma^{-1}) + \alpha(\gamma) = 0$ (cocycle condition). Writing:

$$\alpha(\gamma) = \begin{bmatrix} a_\gamma \\ b_\gamma \end{bmatrix},$$

we obtain

$$q_\gamma \tau + r_\gamma = -(b_\gamma \tau - a_\gamma),$$

or $\beta(\gamma) = \alpha(\gamma)$, as desired. □

Using this we obtain:

(3.10) **Proposition.** *Let $\sigma$ be a section of $X$, represented by the holomorphic function $F$ on $\mathfrak{h}$, and put as usual $G = d^2F/d\tau^2$. Then:*

$$\delta(\sigma) = [\varphi_G]$$

*in $H^1(S, V)$.*

*Proof.* In view of (3.9), we need only check that $\delta(\sigma) = \beta$. Let $\{U_k\}$ be an open covering of $S$ consisting of coordinate discs such that $U_j \cap U_k$ is connected. For each $j$ choose a connected component $\tilde{U}_j$ of $\pi^{-1}(U_j)$. Then if $U_j \cap U_k \neq \emptyset$, there is a unique $\gamma_{jk} \in \Gamma$ with:

$$\tilde{U}_j \cap \gamma_{jk} \tilde{U}_k \neq \emptyset.$$

The group cocycle $\beta$ is sent to the Cech cochain:

$$\hat{\beta}_{jk}=(\pi|_{\tilde{U}_j}^{*})^{-1}\,\beta(\gamma_{jk})|_{U_j\cap U_k}.$$

The class $\delta(\sigma)$ is given as a connecting homomorphism in Cech cohomology (cf. [24, § 3]). One lifts $\sigma$ to $F|_{\tilde{U}_j}$ on $U_j$, representing an element of $\Gamma(U_j,(\mathscr{F}^1)^*)$. Then $\delta(\sigma)$ is given on $U_j\cap U_k$ by the difference, with $\gamma=\gamma_{jk}$,

$$\begin{aligned}&\operatorname{image}\{F(\gamma^{-1}z)M\gamma u_1-F(z)u_1\}\quad\text{in }(\mathscr{F}^1)^*\\&\quad=F(\gamma^{-1}z)j(M\gamma^{-1},\tau)-F(z)\\&\quad=\operatorname{image}\beta(\gamma)=\hat{\beta}_{jk}.\end{aligned}$$

Thus, $\delta(\sigma)=\beta=[\varphi_G]$ by (3.9). □

Let now $\varphi_1,\ \varphi_2\in\Gamma(\bar{S},d\bar{\mathscr{V}}(*D))$, where – very important – we assume $D$ is supported on $S$, and let $\phi_1,\ \phi_2$ be their respective integrals (3.8). We will make use of a fundamental domain $\mathscr{D}$ in $\mathfrak{h}$ for the action of $\Gamma$, of the type defined in [21]. Its most important feature is that its boundary consists of an even number of smooth arcs (edges) with the property that for any edge $\mathscr{E}$ of the oriented boundary $\partial\mathscr{D}$, there exists a unique $\gamma\in\Gamma$ so that $-\gamma(\mathscr{E})$ is also an edge. Thus we have:

(3.11) **Lemma.** *Let $\psi$ be a $\Gamma$-invariant 1-form on $\mathfrak{h}$, integrable on $\partial\mathscr{D}$. Then:*

$$\int_{\partial\mathscr{D}}\psi=0.\quad\square$$

Given $\varphi_1$ and $\varphi_2$ as above, the *Eichler bilinear form* [4] is defined as:

$$I(\varphi_1,\varphi_2)=\int_{\partial\mathscr{D}}{}^t\phi_1\wedge\Theta\varphi_2,$$

where we arrange by choice of $\mathscr{D}$ that $\varphi_1,\ \varphi_2$ are regular on $\partial\mathscr{D}$. If $\varphi_i=\varphi_{G_i}$ with $G_i=d^2F_i/d\tau^2$, then:

$$I(\varphi_1,\varphi_2)=\int_{\partial\mathscr{D}}F_1G_2\,d\tau=\int_{\partial\mathscr{D}}F_2G_1\,d\tau,$$

generalizing the definition given in [3].

(3.12) **Proposition.** $I(\varphi_1,\varphi_2)=[\varphi_1]\cup[\varphi_2]$, *where the cup product:*

$$H^1(\bar{S},j_*V)\times H^1(\bar{S},j_*V)\to H^2(\bar{S},\mathbf{C}).$$

*is induced from the cup product on $V$.*

*Proof.* Both $\phi_j$ and $\varphi_j$ are regular in $\bar{\mathscr{V}}$ at the cusps. To remove their poles on $D$, one makes a correction near $|D|$, obtaining:

$$\eta_j=\varphi_j-d\mu_j,$$

where $\eta_j$ is now a $C^\infty$ 1-form in $\bar{\mathscr{V}}$. Then:

$$
\begin{aligned}
[\varphi_1] \cup [\varphi_2] &= \int_S {}^t\eta_1 \wedge \Theta\eta_2 \\
&= \int_{\mathscr{D}} {}^t\eta_1 \wedge \Theta\eta_2 \qquad \text{(abusing notation)} \\
&= \int_{\mathscr{D}} {}^t d\lambda_1 \wedge \Theta\eta_2 \quad \text{write} \quad \eta_j = d\lambda_j \text{ on } \mathfrak{h} \\
&= \int_{\partial\mathscr{D}} {}^t\lambda_1 \wedge \Theta\eta_2 \\
&= \int_{\partial\mathscr{D}} {}^t\phi_1 \wedge \Theta\eta_2 - \int_{\partial\mathscr{D}} (\mu_1 + v_1) \wedge \Theta\eta_2 \\
&\qquad\qquad\qquad\qquad \phi_1 - \lambda_1 + \mu_1 = v_1, \quad \text{a constant} \\
&= \int_{\partial\mathscr{D}} {}^t\phi_1 \wedge \Theta\eta_2 \qquad \text{(Stokes' theorem and (3.11))} \\
&= \int_{\partial\mathscr{D}} {}^t\phi_1 \wedge \Theta\eta_2 - \int_{\partial\mathscr{D}} {}^t\phi_1 \wedge \Theta(d\mu_2) \\
&= I(\varphi_1, \varphi_2) - \int_{\partial\mathscr{D}} d({}^t\phi_1 \wedge \Theta\mu_2) + \int_{\partial\mathscr{D}} \varphi_1 \wedge \Theta\mu_2 \\
&= I(\varphi_1, \varphi_2);
\end{aligned}
$$

the above integrals all converge because the integrands have at worst logarithmic singularities at the cusps, as follows from the properties of $\bar{\mathscr{V}}$. □

Putting (3.10) and (3.12) together, we have shown that the Eichler pairing of automorphic forms coming from sections of an elliptic surface – as studied by Hoyt and Schwartz ([4, 5, 18]) – are computable in an elementary manner using intersection numbers, as described in §§1, 2. It deserves to be repeated that in (3.12) we (and they) assume that $\tau$ is unramified at the cusps, lest the Eichler pairing be undefined.

*C.* We will develop the Hodge theory necessary to give a unified treatment of several known or conjectured results concerning parabolic cohomology and automorphic forms. We begin by quoting the relevant facts from [25], always taking $V$ to be $R^1 f_* \mathbf{C}$ for an elliptic surface.

To begin, we know that the complexes (3.3) and (3.4) resolve $j_* V$. The Hodge theory for $H^i = H^i(\bar{S}, j_* V)$ can be described by a decreasing filtration $F$ on either of the complexes (call it $K^\cdot$):

$$K^\cdot = F^0 \supset F^1 \supset F^2 \supset F^3 = 0,$$

with successive quotients $\mathrm{Gr}_F^p = F^p/F^{p+1}$, so that the induced filtration $\{F^p H^i\}$ on cohomology gives a Hodge structure of weight $i+1$. Deligne calls this kind of data a *cohomological Hodge complex* of weight one; specifically, it requires:

(3.13) (i) The spectral sequence

$$E_1^{p,q} = \mathbf{H}^{p+q}(\bar{S}, \mathrm{Gr}_F^p K^\cdot) \Rightarrow \mathbf{H}^{p+q}(\bar{S}, K^\cdot) = H^{p+q}$$

degenerates at $E_1$. Then

(a) $\mathbf{H}^i(\bar{S}, F^p K^\cdot)$ maps injectively into $H^i$ (with image $F^p H^i$, of course);

(b) There is a canonical identification of $\mathrm{Gr}_F^p H^i$ with $\mathbf{H}^i(\bar{S}, \mathrm{Gr}_F^p K^\cdot)$.

(ii) $H^{p,q}=F^p H^i \cap \overline{F^q H^i}$ projects isomorphically onto $\mathrm{Gr}_F^p H^i$ for $p+q=i+1$.

The main result of [25] gives the existence of a filtration on the complex (3.3), making it a cohomological Hodge complex for $j_* V$, such that it yields the same Hodge structure on $H^i$ as the one induced by the classical Hodge theory on $\bar{X}$. Let $\bar{\mathscr{F}}^1$ be the sub-bundle of $\bar{\mathscr{V}}$ determined by $\mathscr{F}^1$. Denote by $\Sigma'$ the subset of $\Sigma$ consisting of points where the monodromy is not unipotent; thus $s \in \Sigma'$ if and only if $X_s$ is singular and not of type $\mathrm{I}_b$. The filtration on (3.3) is (see [25, §9]):

$$\text{(3.14)} \quad \begin{aligned} &F^2\colon\ 0 \to \Omega_S^1(\log \Sigma') \otimes \bar{\mathscr{F}}^1 \\ &F^1\colon\ \bar{\mathscr{F}}^1 \to d\bar{\mathscr{V}} \\ &F^0\colon\ \bar{\mathscr{V}} \to d\bar{\mathscr{V}} \end{aligned}$$

hence we have:

$$\text{(3.15)} \quad \begin{aligned} &\mathrm{Gr}_F^1\colon\ \bar{\mathscr{F}}^1 \to \Omega_S^1(\log \Sigma) \otimes \overline{\mathscr{G}\imath}^0 \\ &\mathrm{Gr}_F^0\colon\ \overline{\mathscr{G}\imath}^0 \to 0 \end{aligned}$$

where $\overline{\mathscr{G}\imath}^0$ denotes the quotient $\bar{\mathscr{V}}/\bar{\mathscr{F}}^1$.

Before proceeding further, we need to discuss the ramification of $\tau$ on $\bar{S}$. At a point $s \in S$, where the fiber is smooth, we have the usual notion; in terms of a local parameter $t$ centered at $s$,

$$\tau = \tau(0) + u\, t^m, \quad \text{where} \quad u(0) \neq 0,$$

which we write as:

$$\begin{aligned} \tau &= \tau(0) + (t^m); \\ d\tau &= (t^{m-1})\, dt. \end{aligned}$$

For the singular fibers, there are the following formulae [8, §8] in a suitably chosen local parameter:

| Fiber type | $\tau$ | | $d\tau$ | |
|---|---|---|---|---|
| $\mathrm{I}_0^*$ | $\tau_0 + (t^m)$ | | $(t^{m-1})\, dt$ | |
| $\mathrm{I}_b$, $\mathrm{I}_b^*$ $(b>0)$ | $\dfrac{1}{2\pi i} \log t$ | | $(1)\, dt/t$ | (3.16) |
| II, IV* | $\eta + (t^{h/3})$ | $\eta = e^{2\pi i/3}$, $h \equiv 1(3)$ | $(t^{h/3})\, dt/t$ | |
| II*, IV | $\eta + (t^{h/3})$ | $h \equiv 2(3)$ | $(t^{h/3})\, dt/t$ | |
| III, III* | $i + (t^{h/2})$ | $h \equiv 1(2)$ | $(t^{h/2})\, dt/t$ | |

As a first guess, ramification should indicate excessive vanishing for $d\tau$. However, we will soon see that we must alter this preconception slightly in order to obtain the proper notion. What we do is to look at the size of the cokernel of $d$ in the complex $\mathrm{Gr}_F^1(\bar{\mathscr{V}} \to d\bar{\mathscr{V}})$ (3.15). Computing it is a relatively easy matter, for in $\mathrm{Gr}_F^1$ $d$ becomes $\mathcal{O}_S$-linear, so it suffices to differentiate a generator $\bar{\omega}$ of $\bar{\mathscr{F}}^1$. At a regular fiber, $\bar{\mathscr{F}}^1$ is generated by $\omega=\begin{bmatrix}\tau\\1\end{bmatrix}$, so

$$d(\omega)=(t^{m-1})\,dt\begin{bmatrix}1\\0\end{bmatrix},$$

so $\mathrm{ord}_s\, d(\omega)$ and $\mathrm{ord}_s(d\tau)$ coincide.

(3.17) *Definition.* If $s\in S$, the *order of ramification* of $\tau$ at $s$ is:

$$\mu=\mu(s)=m=1+\dim[(\Omega_S^1\otimes\overline{\mathscr{G}\imath^0})/d\bar{\mathscr{F}}^1]_s;$$

if $s\in\Sigma$, then set:

$$\mu=\dim[(\Omega_S^1(\log\Sigma)\otimes\overline{\mathscr{G}\imath^0})/d\bar{\mathscr{F}}^1]_s,$$

where we are introducing the notation:

$$[\mathscr{S}]_s=\mathscr{S}\otimes\mathcal{O}_s$$

for $\mathcal{O}_S$-modules $\mathscr{S}$.

Note that it is possible to compute $\mu(s)$ by working at any pre-image of $s$ in $\mathfrak{h}$. In order to calculate the ramification orders at points of $\Sigma$, we must determine a generator $\bar{\omega}$ for each of the possible fiber types. The following chart can be verified, using the same notation as before:

| | Type | Monodromy | Eigenvalue/ Eigenvector pairs | $\bar{\omega}$ | |
|---|---|---|---|---|---|
| 1. | $\mathrm{I}_0^*$ | $-\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | | $t^{1/2}\begin{bmatrix}\tau\\1\end{bmatrix}$ | |
| 2. | $\mathrm{I}_b$ | $\begin{pmatrix}1&b\\0&1\end{pmatrix}$ | | $\begin{bmatrix}\tau\\1\end{bmatrix}$ | |
| 3. | $\mathrm{I}_b^*$ $(b>0)$ | $-\begin{pmatrix}1&b\\0&1\end{pmatrix}$ | | $t^{1/2}\begin{bmatrix}\tau\\1\end{bmatrix}$ | |
| 4. | II | $\begin{pmatrix}1&1\\-1&0\end{pmatrix}$ | $-\eta,\ A=\begin{bmatrix}\eta\\1\end{bmatrix};\ -\bar{\eta},\ B=\begin{bmatrix}\bar{\eta}\\1\end{bmatrix}$ | $t^{5/6}[A-t^{h/3}B]$ | (3.18) |
| | II* | $\begin{pmatrix}0&-1\\1&1\end{pmatrix}$ | $-\eta,\ A;\ -\eta,\ B$ | $t^{1/6}[A-t^{h/3}B]$ | |
| | IV | $\begin{pmatrix}0&1\\-1&-1\end{pmatrix}$ | $\bar{\eta},\ A;\ \eta,\ B$ | $t^{2/3}[A-t^{h/3}B]$ | |
| | IV* | $\begin{pmatrix}-1&-1\\1&0\end{pmatrix}$ | $\eta,\ A;\ \bar{\eta},\ B$ | $t^{1/3}[A-t^{h/3}B]$ | |
| | III | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ | $-i,\ \hat{A}=\begin{bmatrix}i\\1\end{bmatrix};\ i,\ \hat{B}=\begin{bmatrix}-i\\1\end{bmatrix}$ | $t^{3/4}[\hat{A}-t^{h/2}\hat{B}]$ | |
| | III* | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ | $i,\ \hat{A};\ -i,\ \hat{B}$ | $t^{1/4}[\hat{A}-t^{h/2}\hat{B}]$ | |

Thus, we obtain for the four sub-divisions above:

(3.19) 1. $d(\bar{\omega})=t^{1/2}\begin{bmatrix}1\\0\end{bmatrix}d\tau=t^m\begin{bmatrix}1\\0\end{bmatrix}dt/t, \quad \mu=m>0$

2. $d(\bar{\omega})=\begin{bmatrix}1\\0\end{bmatrix}dt/t, \quad \mu=0$

3. $d(\bar{\omega})\equiv t^{1/2}\begin{bmatrix}1\\0\end{bmatrix}dt/t, \quad (\operatorname{mod}\bar{\mathscr{F}}^1), \quad \mu=0$

4. In each of the six types, we have

$$\bar{\omega}=t^{\alpha}[v-t^{\beta}\bar{v}]$$

for suitable $\alpha$, $\beta$, and $v$. Then

$$\begin{aligned}d(\bar{\omega})&\equiv-(\alpha+\beta)\,t^{\alpha+\beta}\,\bar{v}\,dt/t\\&=t^{2\alpha+\beta-1}(t^{1-\alpha}\bar{v})\,dt/t.\end{aligned}$$

Thus, $\mu=2\alpha+\beta-1$. If $\beta$ takes on the minimum possible value (1/3, 2/3, or 1/2), then $\mu$ is either 0 or 1, depending on whether $\alpha<1/2$ or $\alpha>1/2$ respectively ("star" or "non-star" fiber types).

We now begin to describe the Hodge structure on $H^1=H^1(\bar{S},j_*V)$.

(3.20) **Proposition**

(1) $F^2H^1\simeq\{G\in A_3(M,\tau)$: *$\varphi_G$ is a holomorphic section of* $\Omega^1_{\bar{S}}(\log\Sigma')\otimes\bar{\mathscr{F}}^1\}$.

(2) $h^{1,1}=\dim H^{1,1}=\sum_{s\in\Sigma}\mu(s)+\sum_{s\in S}[\mu(s)-1]$.

*Proof.* As $F^2H^1\simeq H^1(\bar{S},F^2K^\cdot)\simeq H^0(\bar{S},\Omega^1_{\bar{S}}(\log\Sigma')\otimes\bar{\mathscr{F}}^1)$, (1) follows immediately. For (2), we have:

$$H^{1,1}\simeq\mathbf{H}^1(\bar{S},\operatorname{Gr}^1_F K^\cdot).$$

Let $C$ be the cokernel of $d$ in $\operatorname{Gr}^1_F K^\cdot$. Writing $\operatorname{Gr}^1_F K^\cdot$ as $A\to B$, we have a short exact sequence of complexes:

$$0\to(A\to\operatorname{im}d)\to(A\to B)\to(0\to C)\to 0.$$

As the first term is acyclic, we obtain:

$$H^{1,1}\simeq\mathbf{H}^1(\bar{S},(0\to C))\simeq H^0(\bar{S},C),$$

and thus:

$$h^{1,1}=\dim H^0(\bar{S},C)=\sum_{s\in\Sigma}\mu(s)+\sum_{s\in S}[\mu(s)-1].\quad\square$$

As an immediate consequence of (3.20), we obtain a conceptually clear proof of a result in [23] attributed to Kodaira:

(3.21) **Corollary**

$$h^{1,1} \geqq v(\mathrm{I}_0^*) + v(\mathrm{II}) + v(\mathrm{III}) + v(\mathrm{IV}).$$

*Proof.* One sees from (3.19) that at each $s \in \Sigma$, where a fiber of type $\mathrm{I}_0^*$, II, III, or IV occurs, $\mu(s) \geqq 1$. □

If $\sigma \in \mathfrak{S}$, its image under $\delta$ is a class of type $(1,1)$ in $H^1(\bar{S}, j_* V)$. This follows immediately from the fact that $\delta(\sigma)$ is the reduction of the fundamental class of an algebraic cycle (1.6) – which is of type $(1,1)$ in $H^2(\bar{X})$ – since the Hodge structure is compatible with the Leray spectral sequence. When $p_g(\bar{X}) = 0$, then $H^1(\bar{S}, j_* V)$ is purely of type $(1,1)$; by the Hodge conjecture for surfaces (Lefschetz existence theorem), it is generated by the image of $\delta$, so (3.21) gives a lower bound for the rank of $\mathfrak{S}$.

When $\bar{X}$ is an elliptic modular surface, i.e., if $\tau(z) = z$, we obtain examples of the Hodge structures of Shimura (see [25, §12]). As it is described in [22], the projection:

$$S_3(\Gamma) \to H_P^1(\Gamma, \tilde{V}) \xrightarrow{\mathrm{Re}} H_P^1(\Gamma, \tilde{V}_{\mathbf{R}})$$

is an isomorphism; here, the space of cusp forms $S_3(\Gamma)$ gives $H^{2,0}$, and it is clear that $H^{1,1} = 0$.

We can also give a direct discussion of the following formula given by Shioda:

(3.22) **Proposition** [23]. *Let*

$r = $ *rank of* $\mathfrak{S}$

$g = $ *genus of* $\bar{S}$

$v = $ *number of singular fibers*

$v_1 = $ *number of fibers of type* $\mathrm{I}_b$

$p_g = $ *geometric genus of* $\bar{X}$.

*Then* $r \leqq 4g - 4 + 2v - v_1 - 2p_g$.

*Proof.* Since the homomorphism $\delta$ embeds $\mathfrak{S}/\mathfrak{S}_{\mathrm{tor}}$ in $H^{1,1}$, it suffices to show that the right-hand side of the inequality is equal to $h^{1,1}$. Now:

$$\begin{aligned} h^{1,1} &= \dim H^{1,1} = \dim H^1 - 2\dim H^{2,0} \\ &= \dim H^1 - 2p_g \end{aligned}$$

because the Hodge structure is compatible with the Leray spectral sequence, and those on $H^0(\bar{S}, R^2\bar{f}_* \mathbf{C})$ and $H^2(\bar{S}, R^0\bar{f}_* \mathbf{C})$ are purely of type $(1,1)$. Then using the exact sequence:

$$0 \to H^1 \to H^1(S, V) \to H^0(\bar{S}, R^1 j_* V) \to H^2 = 0$$

and the facts:

$$H^0(\bar{S}, R^1 j_* V) \simeq \bigoplus_{s \in \Sigma} (V/N_s V),$$

where

$$N_s = M(\gamma_s) - I$$

$$\gamma_s = \text{parabolic element associated to } s \in \Sigma,$$

$$H^0 = H^0(S, V) = 0 = H^2(S, V),$$

we obtain:

$$\begin{aligned}\dim H^{1,1} &= \dim H^1(S, V) - \sum_{s \in \Sigma} \dim(V/N_s V) - 2p_g \\ &= -\chi(S) \dim \tilde{V} - v_1 - 2p_g \\ &= -2[\chi(\bar{S}) - v] - v_1 - 2p_g \\ &= 4g - 4 + 2v - v_1 - 2p_g. \quad \square\end{aligned}$$

(3.23) *Remark.* When $p_g = 0$, equality holds, for then $r = h^{1,1}$, as was remarked earlier.

The remainder of the paper will be devoted to the verification of the following result:

(3.24) **Theorem.** *Let $\{s_k\}$ be the points of $\bar{S}$ where $\tau$ is ramified. Put:*

$$D = \sum_k \mu(s_k) \cdot [s_k].$$

*Then, using differential forms of the second kind, $F^1 H^1$ is isomorphic to:*

$$\{\varphi \in H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma') \otimes \bar{\mathscr{F}}^1(D)) : \varphi \text{ is of the second kind}\}.$$

[*Such $\varphi$ are of the form $\varphi_G$, where $G \in A_3(M, \tau)$ may have a pole of order $2\mu(s_k) - 1$ at $s_k \in S$. Also $G$ satisfies the cusp condition at the points of $\Sigma$ with singular fibers of types $\mathrm{I}_b$ or $\mathrm{I}_b^*$ $(b > 0)$ (see (3.19)).*]

W. Hoyt conjectured some cases of this theorem, on the grounds that the dimensions were equal.

Since $F^1 H^1$ contains the image of $\delta$, we would expect, because of (3.10), that if $\varphi$ comes from a section $\sigma$ of the elliptic surface, then it lies in subspace appearing in (3.24). This is easy to check. We may assume that the section passes through the identity components of the singular fibers. Since the construction of $\varphi$ involves taking two $\tau$-derivatives of the function $F$ associated to $\sigma$ (see §3A), we may freely alter $F$ by periods. Since $\sigma$ can be expressed as a local single-valued holomorphic function in terms of a generator of $\bar{\mathscr{F}}^{1*}$, where the periods become multiplied by $t^{\alpha-1}$ for some $0 < \alpha \leqq 1$, in all cases $F$ may be written in the form:

$$F = t^{1-\alpha} H(t) \qquad 0 < \alpha \leqq 1,$$

where $H(t)$ is holomorphic.

Considering the issue case by case, one gets:

| Type | $F$ | $\tau$ | $dF/d\tau$ | $\varphi = d(dF/d\tau) \otimes \begin{bmatrix}\tau\\1\end{bmatrix}$ |
|---|---|---|---|---|
| Non-singular | (1) | $\tau_0 + (t^m)$ | $(t^{1-m})$ | $(t^{-m}) \begin{bmatrix}\tau\\1\end{bmatrix} dt$ |
| $\mathrm{I}_0^*$ | $(t^{1/2})$ | $\tau_0 + (t^m)$ | $(t^{1/2-m})$ | $(t^{-m})\, t^{1/2} \begin{bmatrix}\tau\\1\end{bmatrix} dt/t$ |
| $\mathrm{I}_b$ | (1) | $\frac{b}{2\pi i} \log t$ | $(t)$ | $(1) \begin{bmatrix}\tau\\1\end{bmatrix} dt$ |
| $\mathrm{I}_b^*$ $(b>0)$ | $(t^{1/2})$ | $\frac{b}{2\pi i} \log t$ | $(t^{1/2})$ | $(t^{1/2}) \begin{bmatrix}\tau\\1\end{bmatrix} dt/t$ |
| Other | $(t^{1-\alpha})$ | $\tau_0 + (t^\beta)$ | $(t^{1-\alpha-\beta})$ | $(t^{1-2\alpha-\beta})\, t^\alpha \begin{bmatrix}\tau\\1\end{bmatrix} dt/t$ |

The assertion now follows by comparison with our previous calculation of $\mu$ (3.19).

The proof of (3.24) depends on putting a Hodge filtration on the complex (3.4) compatible with the one given in (3.14). In other terms we will show that there is a *filtered quasi-isomorphism* between the two complexes. Let $K_1$ and $K_2$ be filtered complexes of sheaves with filtrations denoted by $F$. Then a morphism:

$$\iota\colon K_1^{\boldsymbol{\cdot}} \to K_2^{\boldsymbol{\cdot}}$$

is called a filtered quasi-isomorphism if for all $p$:

(3.25) $\quad \mathrm{Gr}_F^p \iota\colon \mathrm{Gr}_F^p K_1^{\boldsymbol{\cdot}} \to \mathrm{Gr}_F^p K_2^{\boldsymbol{\cdot}}$

induces an isomorphism on cohomology sheaves. In the present context, $K_1^{\boldsymbol{\cdot}}$ will be a subcomplex of $K_2^{\boldsymbol{\cdot}}$ with the induced filtration, in which case it suffices to show that, for all $p$, the quotient

$$\mathrm{Gr}_F^p K_2^{\boldsymbol{\cdot}} / \mathrm{Gr}_F^p K_1^{\boldsymbol{\cdot}}$$

is acyclic. A filtered quasi-isomorphism induces an isomorphism on hypercohomology and on all filtration levels thereof.

To motivate the Hodge filtration on (3.4), we will introduce the Hodge filtration for the *mixed* Hodge theory for $\bar{S} - |D|$:

(3.26) **Proposition** [25, §13]. *$H^*(\bar{S} - |D|, j_* V)$ is computable as the hypercohomology of the complex:*

$$L^{\boldsymbol{\cdot}}\colon \bar{\mathscr{V}} \to d\bar{\mathscr{V}}(\log D).$$

*Moreover, the filtration:*

$$F^2\colon\ 0 \to \Omega^1_{\bar{S}}(\log D) \otimes \bar{\mathscr{F}}^1$$

$$F^1\colon\ \bar{\mathscr{F}}^1 \to d\bar{\mathscr{V}}(\log D)$$

$$F^0\colon\ \bar{\mathscr{V}} \to d\bar{\mathscr{V}}(\log D)$$

*induces the Hodge filtration of the natural mixed Hodge structure on the above cohomology groups.* □

We first discuss the case where $\tau$ is unramified at the cusps. The complexes to be defined differ only on $|D|$ from $L^{\bullet}$ above. Since the definition of the filtrations are determined locally, as well as the quasi-isomorphism of filtered complexes, we may assume without loss of generality that $D$ consists of one point $s$ where $\tau$ is ramified to order $m$, and $\bar{S}$ is a disc centered at $t=0$.

We first introduce the usual order-of-pole filtration (cf. [2, p. 80]) on:

$$P^{\bullet} = \Omega^{\bullet}_{S}(*D) \otimes \mathscr{V}$$

(induced by that of $\Omega^{\bullet}_{X}(*f^{-1}(D))$):

$$F^2\colon\ 0 \to \Omega^1_S(1) \otimes \mathscr{F}^1,$$

$$F^1\colon\ \mathscr{F}^1 \to (\Omega^1_S(1) \otimes \mathscr{V}) + (\Omega^1_S(2) \otimes \mathscr{F}^1)$$

for $k \geqq 0$

$$F^{-k}\colon\ \mathscr{V}(k+1) + \mathscr{F}^1(k+2) \to (\Omega^1_S(k+2) \otimes \mathscr{V}) + (\Omega^1_S(k+3) \otimes \mathscr{F}^1)$$

where for a locally-free sheaf $\mathscr{S}$, $\mathscr{S}(n)$ means that one allows poles of order $n$ on $D$. Actually, the infinite length of the filtration can be circumvented, for the inclusion of $F^0 P^{\bullet}$ in $P^{\bullet}$ is a quasi-isomorphism, as will follow. We compute:

$$\mathrm{Gr}^1_F L^{\bullet}\colon\ \mathscr{F}^1 \to \Omega^1_S(1) \otimes \mathscr{G}\imath^0$$

$$\mathrm{Gr}^0_F L^{\bullet}\colon\ \mathscr{G}\imath^0 \to 0$$

$$\mathrm{Gr}^1_F P^{\bullet}\colon\ \mathscr{F}^1(1) \to (\Omega^1_S(1) \otimes \mathscr{G}\imath^0) \oplus [t^{-2}\, dt\, \mathscr{F}^1]_0$$

$$\mathrm{Gr}^0_F P^{\bullet}\colon\ \mathscr{G}\imath^0(1) \oplus [t^{-2} \mathscr{F}^1]_0 \to [t^{-2}\, dt\, \mathscr{G}\imath^0]_0 \oplus [t^{-3}\, dt\, \mathscr{F}^1]_0$$

for $k>0$

$$\mathrm{Gr}^{-k}_F P^{\bullet}\colon\ [t^{-k-1} \mathscr{G}\imath^0]_0 \oplus [t^{-k-2} \mathscr{F}^1]_0 \to [t^{-k-2}\, dt\, \mathscr{G}\imath^0]_0 \oplus [t^{-k-3}\, dt\, \mathscr{F}^1]_0.$$

That $(L^{\bullet}, F) \subset (P^{\bullet}, F)$ is a filtered quasi-isomorphism follows from the elementary fact that $d(t^{-n}) = -n t^{-n-1}\, dt$ which translates cohomologically into the assertion that:

$$[t^{-n} \mathscr{S}]_0 \to [t^{-n-1}\, dt\, \mathscr{S}]_0$$

is acyclic; for instance, there is an exact sequence of complexes (which is not, however, split):

$$0\to([t^{-k-1}\mathcal{G}\imath^0]_0\to[t^{-k-2}\,dt\,\mathcal{G}\imath^0]_0)$$
$$\to(\mathrm{Gr}_F^{-k}P^\cdot)\to([t^{-k-2}\mathcal{F}^1]_0\to[t^{-k-3}\,dt\,\mathcal{F}^1]_0)\to 0$$

so $\mathrm{Gr}_F^{-k}P^\cdot$ is acyclic for $k>0$.

The above is quite general, for it does not make use of the fact that $D$ is the ramification divisor for $\tau$. We have $\mathcal{F}^1$ generated by the vector $\omega=\begin{bmatrix}\tau\\1\end{bmatrix}$, and:

$$d(\omega)=\begin{bmatrix}1\\0\end{bmatrix}d\tau=\begin{bmatrix}1\\0\end{bmatrix}u\,t^{m-1}\,dt$$

for some invertible power series $u(t)$. Thus, we may improve upon the order of pole designations. Define:

$$\tilde{F}^2:\ 0\to\Omega_S^1(1)\otimes\mathcal{F}^1,$$
$$\tilde{F}^1:\ \mathcal{F}^1(m-1)\to(\Omega_S^1(m)\otimes\mathcal{F}^1)+(\Omega_S^1(1)\otimes\mathcal{V}),$$
$$\tilde{F}^0:\ \mathcal{F}^1(m)+\mathcal{V}(1)\to(\Omega^1(m+1)\otimes\mathcal{F}^1)+(\Omega_S^1(2)\otimes\mathcal{V}).$$

(3.27) **Proposition.** *$(P^\cdot,F)\subset(P^\cdot,\tilde{F})$ is a filtered quasi-isomorphism.*

*Proof.* We have:

$$\mathrm{Gr}_{\tilde{F}}^1P^\cdot:\ \mathcal{F}^1(m-1)\to([\Omega_S^1(m)/\Omega_S^1(1)]\otimes\mathcal{F}^1)\oplus(\Omega_S^1(1)\otimes\mathcal{G}\imath^0)$$
$$\mathrm{Gr}_{\tilde{F}}^0P^\cdot:\ [t^{-m}\mathcal{F}^1]_0\oplus\mathcal{G}\imath^0(1)\to[t^{-m-1}\,dt\,\mathcal{F}^1]_0\oplus[t^{-2}\,dt\,\mathcal{G}\imath^0]_0$$

for $k>0$

$$\mathrm{Gr}_{\tilde{F}}^{-k}:\ [t^{-m-k}\mathcal{F}^1]_0\oplus[t^{-k-1}\mathcal{G}\imath^0]_0\to[t^{-m-k-1}\,dt\,\mathcal{F}^1]_0\oplus[t^{-k-2}\,dt\,\mathcal{G}\imath^0]_0.$$

Comparing $\mathrm{Gr}_{\tilde{F}}^pP^\cdot$ with $\mathrm{Gr}_F^pP^\cdot$, the result follows by elementary considerations.

It remains now to push the notion of "second kind" (sk) through the filtered complexes we have just defined. We obtain in this manner:

$$(L^\cdot_{\mathrm{sk}},F)=(\Omega^\cdot_S(V),F)$$
$$(P^\cdot_{\mathrm{sk}},F)$$
$$(P^\cdot_{\mathrm{sk}},\tilde{F}).$$

By almost exactly the same arguments as were used in proving (3.27), we may deduce:

(3.28) **Proposition.** *The inclusions*

$$(\Omega^\cdot_S(V),F)\subset(P^\cdot_{\mathrm{sk}},F)\subset(P^\cdot_{\mathrm{sk}},\tilde{F})$$

*are filtered quasi-isomorphisms.* □

Using (3.28), we can now prove (3.24). We have:

$$\mathrm{Gr}_F^1 P_{\mathrm{sk}}^{\bullet}\colon \mathscr{F}^1(m-1) \to ([\Omega_S^1(m)_{\mathrm{sk}}/\Omega_S^1] \otimes \mathscr{F}^1) \oplus (\Omega_S^1 \otimes \mathscr{G}\imath^0),$$

which we abbreviate as:

$$G^{\bullet}\colon D \to A/B \oplus C.$$

Since $d(\omega) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$, $d_2\colon D \to C$ is an isomorphism. If we set:

$$G_1^{\bullet}\colon 0 \to A/B$$

$$G_2^{\bullet}\colon D \xrightarrow{d_2} C,$$

then we have a short exact sequence of complexes:

$$0 \to G_1^{\bullet} \to G^{\bullet} \to G_2^{\bullet} \to 0.$$

We should remember at this point that although the definitions of the various complexes is seemingly made on $S$, they carry, by agreement, a prescribed extension to $\bar{S}$ as in (3.3). By construction, $G_2^{\bullet}$ is acyclic, so:

$$\mathbf{H}^*(\bar{S}, G_1^{\bullet}) \xrightarrow{\sim} \mathbf{H}^*(\bar{S}, G^{\bullet}).$$

Therefore:

$$\mathrm{Gr}_F^1 H^1 \simeq \mathbf{H}^1(\bar{S}, G^{\bullet}) \simeq H^0(\bar{S}, A/B).$$

Finally, it follows from the exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow H^0(\bar{S}, B) & \longrightarrow & H^0(\bar{S}, A) & \longrightarrow & H^0(\bar{S}, A/B) & \longrightarrow & H^1(\bar{S}, B) \\ \wr\| & & & & \wr\| & & \wr\| \\ F^2 H^1 & & & & \mathrm{Gr}_F^1 H^1 & \xrightarrow{0} & F^2 H^2 \end{array}$$

that $F^1 H^1 \simeq H^0(\bar{S}, A) = H^0(\bar{S}, \Omega_S^1(D)_{sk} \otimes \bar{\mathscr{F}}^1)$, which is the desired result (3.24) when $|D| \cap \Sigma = \emptyset$.

The proof is nearly identical when $\tau$ has ramification at the cusps. We need only make the appropriate changes in the complex. One obtains for $\tilde{F}^1$ at a point of $\Sigma' \subset \Sigma$ (if a singular fiber is of type $\mathrm{I}_b$, no change from (3.3) is necessary):

$$\tilde{F}^1\colon \bar{\mathscr{F}}^1(\mu) \to [\Omega_S^1(\log \Sigma)] \otimes [\bar{\mathscr{F}}^1(\mu) + \bar{\mathscr{V}}]$$

$$\mathrm{Gr}_{\tilde{F}}^1\colon \bar{\mathscr{F}}^1(\mu) \to [\Omega_S^1(\log \Sigma) \otimes (\bar{\mathscr{F}}^1(\mu)/\bar{\mathscr{F}}^1)] \oplus [\Omega_S^1(\log \Sigma) \otimes \overline{\mathscr{G}\imath^0}].$$

The rest of the argument is essentially the same as before; details are left to the reader. □

(3.29) *Remark.* Let us point out how (3.24) relates to the classical description of $F^1 H^1(\bar{X}, \mathbf{C})$ using meromorphic 2-forms of the second kind. Let $w$ be the variable on $\mathbf{C}$ in the universal cover $\tilde{X} = \mathfrak{h} \times \mathbf{C}$ of $X$. Then, given $G \in A_3(M, \tau)$

such that $[\varphi_G] \in F^1 H^1(\bar{S}, j_* V)$ (cf. (3.24)), $\psi_G = G(z)\, dw \wedge d\tau$ defines a 2-form on $X$ which is of the second kind on $\bar{X}$. It is easy to see that the cohomology class of $\psi_G$ lies in $L^1$ and is equal to $[\varphi_G]$ in $L^1/L^2 \simeq H^1(\bar{S}, j_* V)$.

## References


1. Clemens, C.H.: Degeneration of Kähler manifolds. Duke Math. J. **44**, 215–290 (1977)
2. Deligne, P.: Equations differentielles à points singuliers réguliers. In: Lecture Notes in Math. 163. Berlin-Heidelberg-New York: Springer 1970
3. Eichler, M.: Eine Verallgemeinerung der Abelschen Integrale. Math. Zeitschr. **67**, 267–298 (1957)
4. Hoyt, W.: Parabolic cohomology and cusp forms of the second kind for extensions of the field of modular functions. Preprint 1978
5. Hoyt, W., Schwartz, C.: Period relations for the Weierstrass equation $y^2 = 4x^3 - 3ux - u$. In preparation (1979)
6. Iitaka, S.: Deformations of compact complex surfaces. In: Global analysis, papers in honor of K. Kodaira. Princeton: Princeton University Press 1969
7. Kas, A.: On the deformation types of regular elliptic surfaces. Preprint 1976
8. Kodaira, K.: On compact analytic surfaces, II–III. Annals of Math. **77**, 563–626; **78**, 1–40 (1963)
9. Kodaira, K.: On homotopy $K-3$ surfaces. In: Essays on topology and related topics, Mémoires dédiés à George deRham. Berlin-Heidelberg-New York: Springer 1970
10. Kodaira, K.: On the structure of compact analytic surfaces, I. Amer. J. Math. **87**, 751–798 (1964)
11. Mandelbaum, R.: On the topology of elliptic surfaces. Preprint 1977
12. Mandelbaum, R.: On the topology of non-simply connected elliptic surfaces with degenerate fibers. Manuscript 1978
13. Manin, Ju.I.: The Tate height of points on an abelian variety; its variants and applications. Amer. Math. Soc. Transl. (Series 2) **59**, 82–110 (1966)
14. Néron, A.: Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. Pub. Math. I.H.E.S. **21** (1964)
15. Néron, A.: Quasi-fonctions et hauteurs sur les variétés abéliennes. Annals of Math. **82**, 249–331 (1965)
16. Ogg, A.: Cohomology of abelian varieties over function fields. Annals of Math. **76**, 185–212 (1962)
17. Schwartz, C.: Independent solutions of certain Weierstrass equations. Manuscript 1977
18. Schwartz, C.: On generators of the group of rational solutions of a certain Weierstrass equation. Trans. A.M.S., to appear
19. Shafarevitch, I.: Principal homogeneous spaces defined over a function field. Amer. Math. Soc. Transl. (Series 2) **37**, 85–114 (1964)
20. Shafarevitch, I., and others: Algebraic Surfaces. Moscow: Proc. Steklov Institute 1965; English Translation, Providence: A.M.S. 1967
21. Shimura, G.: Introduction to the arithmetic theory of automorphic forms. Princeton: Princeton University Press, 1971
22. Shimura, G.: Sur les intégrales attachées aux formes automorphes. J. Math. Soc. Japan **11**, 291–311 (1959)
23. Shioda, T.: On elliptic modular surfaces. J. Math. Soc. Japan **24**, 20–59 (1972)
24. Zucker, S.: Generalized intermediate Jacobians and the theorem on normal functions. Inventiones Math. **33**, 185–222 (1976)
25. Zucker, S.: Hodge theory with degenerating coefficients: $L_2$ cohomology in the Poincaré metric, Annals of Math. (1979)


Received July 15, 1978/March 26, 1979