Remarks on Expository Writing in Mathematics

Robert B. Ash

Successful graduate students in mathematics are able to reach an advanced level in one or more areas. Textbooks are an important part of this process. A skilled lecturer is able to illuminate and clarify many ideas, but if the pace of a course is fast enough to allow decent coverage, gaps will inevitably result. Students will depend on the text to fill these gaps, but the experience of most students is that the usual text is difficult for the novice to read. At one extreme, the text is a thousand page, twenty pound encyclopedia which cannot be read linearly in a finite amount of time. At the other extreme, the presentation in the book is essentially a seminar lecture with huge gaps.

So it seems that improvements in readability of textbooks would be highly desirable, and the natural question is "What makes a text readable?" Is it possible to answer such a question concretely? I am going to try.

First, we need to be clear on exactly who is trying to read these books. Textbooks that are opaque for students may turn out to be quite useful to the research specialist. I will assume that the reader of the text is not already an expert in the area.

The path to readability is certainly not unique, but here is some advice that may be useful.

1. Adopt a Linear Style

The idea is to help the student move from point A to point B as quickly and efficiently as possible. When learning a subject, I don't like lengthy detours, digressions or interruptions. I don't like having my path blocked by exercises that are used in the text but are not accompanied by solutions. It is fine to say that the reader should be challenged to participate actively in the learning process, but if I have to do 100 exercises to get through Chapter 1, and I get stuck on exercise 3, the chances are that I will abandon the project. This leads naturally to the next category.

2. Include Solutions to Exercises

There is an enormous increase in content when solutions are included. I trust my readers to decide which barriers they will attempt to leap over and which obstacles they will walk around. This often invites the objection that I am spoon-feeding my readers. My reply is that I would love to be spoon-fed class field theory, if only it were possible. Abstract mathematics is difficult enough without introducing gratuitous roadblocks.

3. Discuss the Intuitive Content of Results

A formal proof of the fundamental theorem of calculus should be accompanied by some intuition: If x changes by a small amount dx, then the area under the curve $y = f(x)$ changes by $dA = f(x)\,dx$, hence $dA/dx = f(x)$. Engineers and physicists will appreciate this viewpoint. They are trying to explain and predict the behavior of physical systems. This is a legitimate use of mathematics, and for this purpose, a formal proof may very well be unnecessary or even counterproductive.

Here is an example at a more advanced level. If $X = \text{Spec } A$ is the set of all prime ideals of the commutative ring A, topologized with the Zariski topology, it is useful to think informally of the elements $f \in A$ as functions on X. The value of f at the point P is the coset $f + P \in A/P$. Thus $f(P) = 0$ iff $f \in P$. This suggests an abstract analog of the zero set of a collection of polynomials. For any subset S of A, we define $V(S) = \{P \in \text{Spec } A : P \supseteq S\}$. In other words, $V(S)$ consists of all $P \in X$ such that every $f \in S$ vanishes at P.

4. Replace Abstract Arguments by Algorithmic Procedures if Possible

One formal proof that a countable union of countable sets A_n is countable goes like this. Let f_n be a surjective mapping of the positive integers N onto A_n. If we define $h : N \times N \to \cup_{n=1}^{\infty} A_n$ by $h(n,m) = f_n(m)$, then h is surjective and therefore the union of the A_n is countable (because $N \times N$ is countable). But a proof that actually gives an algorithm has more impact. List the sets A_n as an array:

$$A_1 : a_{11}\ a_{12}\ a_{13} \cdots$$
$$A_2 : a_{21}\ a_{22}\ a_{23} \cdots$$
$$A_3 : a_{31}\ a_{32}\ a_{33} \cdots$$
$$\vdots$$

Then count the union by Cantor's diagonal method, for example,
$a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, a_{32}, a_{41}$, and so on.

5. Use the Concrete Example with All the Features of the General Case.

Consider the Euclidean algorithm and its corollary that if d is the greatest common divisor of a and b, there exist integers s and t such that $sa + tb = d$. I like to take a numerical example such as $a = 123$ and $b = 54$ and carry out all the details. In this case, a carefully chosen example will have all the features of the general case. What this means is that a formal proof essentially involves substituting Greek letters for numbers. The concrete example instructs you on how to write a formal proof, and is easier to follow.

6. Avoid Serious Gaps in the Reasoning

This may be the most important component of readability. Nothing wears out the reader more than a proof in which step n+1 does not follow from step n. If the conclusions are not justified by the arguments given, the student trying to learn the subject is in for a rough ride.

Gaps can take several forms. An algebraic computation may be difficult to follow because not enough details are provided. A result from a later chapter can be tacitly assumed. Knowledge of a subject not covered in the text might be required. But perhaps the most common gap occurs when steps that are essential for understanding the proof are simply omitted. One can argue that it may be difficult to determine how much detail to give. After all, we are not going to write proofs in the formal language of set theory. We are not going to expand the text to ten thousand pages. We want the steps in a proof to be as clear as possible, and at the same we want the argument to be as brief as possible. My job as expositor is to resolve the conflict between these two objectives. If I go too far in one direction or the other, my readers will let me know about it.

Closing Comments

I hope to see a change in the reward structure and system of values at research-oriented universities so that teaching and expository writing become legitimate as a specialty. This will help to improve the current situation in which many advanced areas of mathematics are inaccessible to most students because no satisfactory exposition exists. I hope to see more mathematicians write lecture notes for their courses and post the results on the web for all to use.