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ON O_{n+1}

H. ARAKI, A. L. CAREY and D. E. EVANS

1. INTRODUCTION

The C^*-algebra O_n generated by n-isometries with orthogonal ranges was introduced by Cuntz in [6] and shown to be simple and independent of the choice of generators. These C^*-algebras (for $2 \leqslant n < \infty$) are closely associated with the full n-shift in topological Markov chain theory [8,9] and are C^*-analogues of factors of type $\mathrm{III}_{1/n}$. They became important initially through providing counterexamples to various questions but subsequently have become interesting C^*-algebras in their own right.

In [9] the third author gave a construction of O_n from annihilation and creation operators on a full Fock space and began an investigation of certain states and automorphisms which are natural analogues of quasifree states and quasifree automorphisms on the CAR algebra. These states and automorphisms on O_n are also termed quasifree. Mainly type I states were analysed in [9] while here we proceed to investigate further properties in the non-type I case.

The properties of quasifree states on O_n are more complex than in the CAR case (for example they need not be factor states) and we concentrate on those which are "compatible" (in a sense made precise below) with the structure of the stable algebra $\mathscr{C}\otimes O_n$ as a C^*-crossed product $C^*\{\overline{F}_n, \mathbf{Z}\}$ of an AF-algebra $\overline{F}_n$ and the shift.

We investigate firstly (in Section 3) the question of when the shift, as an automorphism of $\overline{F}_n$, is extendable to an automorphism of the weak closure, in the representation of $\overline{F}_n$ obtained by restricting a quasifree state of $\mathscr{C}\otimes O_n$. When the shift is implementable, the von Neumann algebra generated by $\mathscr{C}\otimes O_n$ in a quasifree state, is identified with a W^*-crossed product. This identification enables us to give sufficient conditions for a quasifree state to be a factor state. The criterion of primariness in the general situation (Theorem 3.5) is shown by a different method. We then go on, in § 4, to show that the quasifree automorphisms defined in [9] are usually outer (Theorem 4.3). In particular, in the canonical quasifree state on O_n, $2 \leqslant n < \infty$, no quasifree automorphisms are weakly inner while for O_∞ they are uniformly outer but weakly inner. This improves results of [1, 7, 9] where it was shown that every quasifree automorphism of O_n, for $n < \infty$, is not uniformly inner.

2. PRELIMINARIES

If $\mathscr{H}$ is a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ (respectively $\mathscr{K}(\mathscr{H})$, $\mathscr{K}_0(\mathscr{H})$ denotes the bounded (respectively compact, finite rank) operators on $\mathscr{H}$. Then $\widetilde{\mathscr{K}(\mathscr{H})}$ denotes the C^*-subalgebra $\mathscr{K}(\mathscr{H}) + \mathbf{C}1_{\mathscr{H}}$ of $\mathscr{B}(\mathscr{H})$. $\mathscr{T}_1(H)$ denotes the positive trace class operators K on $\mathscr{H}$ such that $\operatorname{tr} K \leqslant 1$, if $\mathscr{H}$ is infinite dimensional and $\operatorname{tr} K = 1$ otherwise. The trace is always normalised so that the trace of a minimal projection is 1. If $K \in \mathscr{T}_1(\mathscr{H})$, let ρ_K denote the normalised state on $\widetilde{\mathscr{K}(\mathscr{H})}$:

$$\rho_K(x + \lambda 1_{\mathscr{H}}) = \operatorname{tr}(Kx) + \lambda, \quad x \in \mathscr{K}(\mathscr{H}), \quad \lambda \in \mathbf{C}. \tag{2.1}$$

$\mathscr{C}$ denotes the compact operators on a fixed infinite dimensional separable Hilbert space.

Let $\mathscr{H}$ be an n-dimensional complex Hilbert space, where $2 \leqslant n \leqslant \infty$, with complete orthonormal basis $\{e_i\}_{i=1}^n$. Then $O(\mathscr{H})$ is the unital C^*-algebra generated by the range of a linear map O defined on $\mathscr{H}$ such that

$$O(h)^* O(k) = \langle k, h \rangle 1, \quad h, k \in \mathscr{H} \tag{2.2}$$

$$\sum_{i=1}^{n} O(e_i)\, O(e_i)^* \leqslant 1 \tag{2.3}$$

with equality holding in (2.3) should n be finite. Then $O(\mathscr{H}) \simeq O_n$, the C^*-algebra of Cuntz.

If $u \in U(\mathscr{H})$, the group of unitaries on $\mathscr{H}$, the quasifree automorphism $O(u)$ is the $*$-automorphism of $O(\mathscr{H})$ such that $O(u)O(h) = O(uh)$, $h \in \mathscr{H}$. In particular let $\alpha_t = O(t)$, $t \in \mathbf{T}$ denote the gauge group. The fixed point algebra $F(\mathscr{H}) = O(\mathscr{H})^\alpha$ is an AF-algebra (UHF if n is finite). $F(\mathscr{H})$ is the C^*-subalgebra of $\bigotimes_{i=1}^{\infty} \widetilde{\mathscr{K}(\mathscr{H})}$ generated by $\bigcup_{r=0}^{\infty} \bigotimes_1^r \mathscr{K}(\mathscr{H})$, and we let $F_0(\mathscr{H})$ denote the $*$-subalgebra generated by $\bigcup_{r=0}^{\infty} \bigotimes_1^r \mathscr{K}_0(\mathscr{H})$ identified with

$$\bigcup_r \operatorname{lin}\{O(h_1) \ldots O(h_r) O(f_r)^* \ldots O(f_1)^* : f_i,\ h_i \in \mathscr{H}\}.$$

Then $F(u) = O(u)|F(\mathscr{H})$ is the restriction of $\bigotimes_1^\infty \mathrm{Ad}(u)$ on $\bigotimes_1^\infty \widetilde{\mathscr{K}(\mathscr{H})}$ to $F(\mathscr{H})$. Let $e = e_1$, and define

$$f_i = \begin{cases} e \otimes \bar{e} & i \leqslant 0 \\ 1_{\mathscr{H}} & i \geqslant 1. \end{cases}$$

Let $\overline{F(\mathscr{H})}$ denote the C^*-subalgebra of the restricted tensor product [2] $\mathfrak{B} = \bigotimes_{i=1}^{\infty} (\widetilde{\mathscr{K}(\mathscr{H})}, f_i)$ generated by $\bigcup_{r\geqslant 0} \bigotimes_{-r}^{r} \mathscr{K}(\mathscr{H})$, and let $f = \bigotimes_{-\infty}^{\infty} f_i$, and let $\overline{F_0(\mathscr{H})}$ be the $*$-subalgebra generated by $\bigcup_{r\geqslant 0} \bigotimes_{-r}^{r} \mathscr{K}_0(\mathscr{H})$. Here $\mathfrak{B} = \lim_{i\to-\infty} \overline{F_i}$, where $\overline{F_i} = \bigotimes_{j=i}^{\infty} \widetilde{\mathscr{K}(\mathscr{H})}$ and $\overline{F_i}$ is embedded in $\overline{F}_{i-1}$ by $x \to f_{i-1} \otimes x$. Let $\mathbf{Z}$ act on $\overline{F(\mathscr{H})}$ by a shift σ to the right, and let $\overline{O(\mathscr{H})}$ denote the corresponding crossed product $C^*(\overline{F(\mathscr{H})}, \mathbf{Z})$. Then

$$(2.4)\ (\overline{O(\mathscr{H})}, f\overline{O(\mathscr{H})}f, f\overline{F(\mathscr{H})}f, \hat{\sigma}) \cong (\mathscr{C} \otimes O(\mathscr{H}), q \otimes O(\mathscr{H}), q \otimes F(\mathscr{H}), \mathrm{id} \otimes \alpha)$$

where $\hat{\sigma}$ denotes the dual action, and q is a fixed minimal projection in $\mathscr{C}$.

If $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$ is a sequence of operators in $\mathscr{T}_1(\mathscr{H})$, let $\rho_{\mathfrak{K}}$ denote the restriction of the product state $\bigotimes_{i=1}^{\infty} \rho_{K_i}$ on $\bigotimes_{i=1}^{\infty} \widetilde{\mathscr{K}(\mathscr{H})}$ to $F(\mathscr{H})$. Let $\bar{P}$ and P denote the canonical projections of $\overline{O(\mathscr{H})}$ and $O(\mathscr{H})$ on $\overline{F(\mathscr{H})}$ and $F(\mathscr{H})$ respectively, so that $\bar{P} = 1 \otimes P$. We let $\omega_{\mathfrak{K}}$ denote the state $\rho_{\mathfrak{K}} \circ P$ on $O(\mathscr{H})$. Such a state is said to be quasi-free. If $\mathfrak{K}$ consists of a constant sequence, $K \in \mathscr{T}_1(\mathscr{H})$ say, we write ω_K for $\omega_{\mathfrak{K}}$, e.g. $K = 1/n \in \mathscr{T}_1(\mathscr{H})$, and $\omega_{1/n}$ is called the canonical state.

Let $\overline{\psi_{\mathfrak{K}}}$ denote the state on $\overline{F(\mathscr{H})}$ obtained by taking the inductive limit of $\bigotimes_i^0 \rho_{e\otimes\bar{e}} \otimes \rho_{\mathfrak{K}}$ on $\overline{F_i} = \bigotimes_i^\infty \widetilde{\mathscr{K}(\mathscr{H})}$ and restricting to $\overline{F(\mathscr{H})}$. Let $\overline{\varphi}_{\mathfrak{K}} = \overline{\psi}_{\mathfrak{K}} \circ \bar{P}$ be the corresponding state on $\overline{O(\mathscr{H})}$. Then under (2.4), $(\overline{\varphi}_{\mathfrak{K}}, \overline{\psi}_{\mathfrak{K}})$ on $(\overline{O(\mathscr{H})}, \overline{F(H)})$ are identified with $(\rho_q \otimes \omega_{\mathfrak{K}}, \rho_q \otimes \rho_{\mathfrak{K}})$ on $(\mathscr{C} \otimes O(\mathscr{H}), \mathscr{C} \otimes F(\mathscr{H}))$.

Let $\bar{\rho}_{\mathfrak{K}}$ denote the weight on $\overline{F(\mathscr{H})}$ obtained by taking the inductive limit of the weights $\bigotimes_i^0 (\mathrm{tr}(\cdot)) \otimes \rho_{\mathfrak{K}}$ on $\bigotimes_i^\infty \widetilde{\mathscr{K}(\mathscr{H})}$ and restricting to $\overline{F(\mathscr{H})}$. Let $\overline{\omega}_{\mathfrak{K}} = \bar{\rho}_{\mathfrak{K}} \circ \bar{P}$ denote the corresponding weight on $\overline{O(\mathscr{H})}$. Then under (2.4), $(\overline{\omega_{\mathfrak{K}}}, \overline{\rho_{\mathfrak{K}}})$ on $(\overline{O(\mathscr{H})}, \overline{F(\mathscr{H})})$ are identified with $(\mathrm{tr} \otimes \omega_{\mathfrak{K}}, \mathrm{tr} \otimes \rho_{\mathfrak{K}})$ on $(\mathscr{C} \otimes O(\mathscr{H}), \mathscr{C} \otimes F(\mathscr{H}))$.

If n is finite, let $\mathbf{Z}_n$ denote the group of integers mod n, and let $\mathbf{Z}$ act on the restricted product group $\bigoplus_{-\infty}^{\infty} \mathbf{Z}_n$ (equipped with the discrete topology) by a shift σ to the right. Then the semidirect product group $G_n = \left(\bigoplus_{-\infty}^{\infty} \mathbf{Z}_n \right) \times_{\sigma} \mathbf{Z}$ acts on $X_n = \left(\bigoplus_{-\infty}^{0} \mathbf{Z}_n \right) \oplus \left(\prod_{1}^{\infty} \mathbf{Z}_n \right)$ (equipped with the inductive limit topology) by translation and finitely many changes of coordinates. We identify $X_n \subset \prod_{-\infty}^{\infty} \mathbf{Z}_n$ as sets. Then

$$(C^*(X_n, G_n), C^*(X_n, \oplus \mathbf{Z}_n)) \simeq (\overline{O(\mathscr{H})}, \overline{F(\mathscr{H})}) \tag{2.5}$$

and the projection f is identified with the characteristic function of $\{x = (x_i) \in X_n : x_i = 0,\ i \leqslant 0\}$.

For each (i, j) $i = 1, 2, \ldots, j \in \mathbf{Z}_n$ let $\varkappa_{ij}$ be a positive real number with $\sum_j \varkappa_{ij} = 1$, and let $\mathfrak{k} = \{\varkappa_{ij}\}_{i,j}$. Let μ_i denote the probability measure on $\mathbf{Z}_n$ given by $\mu_i(j) = \varkappa_{ij}$, and $K_i = \sum_{j=1}^{n} \varkappa_{ij} e_j \otimes \bar{e}_j \in \mathscr{T}_1(\mathscr{H})$, $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$. Let $\mu_{\mathfrak{k}}$ denote the product measure $\prod_{i=1}^{\infty} \mu_i$ on $\prod_{i=1}^{\infty} \mathbf{Z}_n$. Let $\bar{\psi}_{\mathfrak{k}}$ denote the probability measure on X_n obtained by taking the inductive limit of the measures $\prod_i^0 \delta_0 \otimes \mu_{\mathfrak{k}}$ on $\prod_i^{\infty} \mathbf{Z}_n$, where δ_0 is the Dirac point measure at 0. Let $\bar{Q}$ denote the canonical projection of $C^*(X_n, G_n)$ on $C_0(X_n)$ and $\bar{Q}_0$ that of $C^*(X_n, \oplus \mathbf{Z}_n)$ on $C_0(X_n)$. Then under the identification (2.5), $(\bar{\varphi}_{\mathfrak{K}}, \bar{\psi}_{\mathfrak{K}})$ on $(\overline{O(\mathscr{H})}, \overline{F(\mathscr{H})})$ corresponds to $(\bar{\varphi}_{\mathfrak{k}} \circ \bar{Q}, \bar{\psi}_{\mathfrak{k}} \circ \bar{Q}_0)$ on $(C^*(X_n, G_n), C^*(X_n, \oplus \mathbf{Z}_n))$.

Let $\bar{\mu}_{\mathfrak{K}}$ denote the (infinite) measure on X_n obtained by taking the inductive limit of the measures $\left(\prod_i^0 \delta \right) \times \mu_{\mathfrak{k}}$ on $\prod_i^{\infty} \mathbf{Z}_n$, where δ is counting measure on $\mathbf{Z}_n$. Then the weights $(\bar{\mu}_{\mathfrak{K}} \circ \bar{Q}, \bar{\mu}_{\mathfrak{K}} \circ \bar{Q}_0)$ on $(C^*(X_n, G_n), C^*(X_n, \oplus \mathbf{Z}_n))$ are identified with $(\bar{\omega}_{\mathfrak{K}}, \bar{\rho}_{\mathfrak{K}})$ on $(\overline{O(\mathscr{H})}, \overline{F(\mathscr{H})})$.

More details on these identifications may be found in [6] and [9].

3. THE SHIFT

We first consider the question of implementability and extendability of the shift automorphism σ on $\overline{F(\mathscr{H})}$. Let $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$ be a sequence of operators in $\mathscr{T}_1(\mathscr{H})$, with $K_i > 0$. Then $\bar{\rho}_{\mathfrak{K}}$ is a faithful weight on $\overline{F(\mathscr{H})}$, and if $\mathfrak{N}_{\mathfrak{K}} = \{x \in \overline{F(\mathscr{H})} : \bar{\rho}_{\mathfrak{K}}(x^*x) < \infty\}$, there exists an injective map $\Lambda_{\mathfrak{K}}$ from $\mathfrak{N}_{\mathfrak{K}}$ into a Hilbert space $\overline{\mathscr{H}}_{\mathfrak{K}}$ and a representation $\bar{\pi}_{\mathfrak{K}}$ of $\overline{F(\mathscr{H})}$ on $\overline{\mathscr{H}}_{\mathfrak{K}}$ such that

(3.1) $$\langle \Lambda_{\mathfrak{K}}(x), \Lambda_{\mathfrak{K}}(y) \rangle = \bar{\omega}_{\mathfrak{K}}(y^*x), \quad x, y \in \mathfrak{N}_{\mathfrak{K}}$$

(3.2) $$\bar{\pi}_{\mathfrak{K}}(x)\Lambda_{\mathfrak{K}}(y) = \Lambda_{\mathfrak{K}}(xy) \quad x \in \overline{F(\mathscr{H})},\ y \in \mathfrak{N}_{\mathfrak{K}}$$

(3.3) $$\{\Lambda_{\mathfrak{K}}(y) : y \in \mathfrak{N}_{\mathfrak{K}}\}^{-} = \overline{\mathscr{H}}_{\mathfrak{K}}.$$

The triple $(\overline{\mathscr{H}}_{\mathfrak{K}}, \bar{\pi}_{\mathfrak{K}}, \Lambda_{\mathfrak{K}})$ can be identified as follows. Let H_i denote the Hilbert-Schmidt operators on $\mathscr{H}$, for $i \in \mathbf{Z}$ and

$$\Omega_i = K_i^{1/2}\ i \geqslant 1, \ \ \Omega_i = e \otimes \bar{e} \ \text{ if } i \leqslant 0, \text{ so } \Omega_i \in H_i.$$

Under the identification $\overline{O(\mathscr{H})} = \mathscr{K} \otimes O(\mathscr{H})$, $\bar{\omega}_{\mathfrak{K}} = \sum_{i=1}^{\infty} f_i$, where $f_i = \rho_{h_i \otimes \bar{h}_i} \otimes \omega_{\mathfrak{K}}$, where $\{h_i\}_{i=1}^{\infty}$ is a complete orthonormal set in the underlying Hilbert space for $\mathscr{C}$. Then (cf. [4]) $\overline{\mathscr{H}}_{\mathfrak{K}} \simeq \bigoplus_{i=1}^{\infty} \mathscr{H}(f_i)$, where $(\mathscr{H}(f_i), \pi_i, \theta_i)$ is the GNS triplet for the state f_i, and $\Lambda_{\mathfrak{K}}(x)$ corresponds to $\bigoplus_{i=1}^{\infty} (\pi_i(x)\theta_i)$, $x \in \mathfrak{N}_{\mathfrak{K}}$. It is then clear that if $\mathscr{K}_0 = \mathrm{lin}\{h_i \otimes \bar{h}_j\}$, that $\Lambda_{\mathfrak{K}}(\mathscr{K}_0 \otimes F_0(\mathscr{H}))$ is dense in $\overline{\mathscr{H}}_{\mathfrak{K}}$. Then $\mathscr{K}_0 \otimes F_0(\mathscr{H}) \simeq \overline{F_0(\mathscr{H})}$, and define a map

$$\Lambda_0 : \overline{F_0(\mathscr{H})} \to \bigotimes_{-\infty}^{\infty} (H_i, \Omega_i) = \overline{\mathscr{H}}^0_{\mathfrak{K}}$$

by

$$\Lambda_0(x) = \bigotimes_{i<-j_0} \Omega_i \otimes \Big(\bigotimes_{|i|<j_0} y_i\Big) \otimes \Big(\bigotimes_{i>j_0} \Omega_i\Big)$$

if $x = \bigotimes_{-j_0}^{j_0} x_i \in \bigotimes_{-j_0}^{j_0} \mathscr{K}_0(\mathscr{H})$, and $y_i = x_i\Omega_i\ i \geqslant 1$, $y_i = x_i$, $i \leqslant 0$. It is clear that there exists a representation $\bar{\pi}^0_{\mathfrak{K}}$ of $\overline{F(\mathscr{H})}$ on $\bar{\mathscr{H}}^0_{\mathfrak{K}}$ such that $\bar{\pi}_0(x)\Lambda_0(y) = \Lambda_0(xy)$ if $x, y \in \overline{F_0(\mathscr{H})}$. Moreover $\langle \Lambda_0(x), \Lambda_0(y)\rangle = \bar{\rho}_{\mathfrak{K}}(y^*x) = \langle \Lambda_{\mathfrak{K}}(x), \Lambda_{\mathfrak{K}}(y)\rangle$ $x, y \in \overline{F_0(\mathscr{H})}$. Hence there exists a unitary U of $\bar{\mathscr{H}}^0_{\mathfrak{K}} = [\Lambda_0(\overline{F_0(\mathscr{H})})]^-$ onto $\mathscr{H}_{\mathfrak{K}} = [\Lambda_{\mathfrak{K}}(\overline{F_0(\mathscr{H})})]^-$ such that $U\Lambda_0(x) = \Lambda_{\mathfrak{K}}(x)$, $x \in \overline{F_0(\mathscr{H})}$. In this way we can identify $(\bar{\mathscr{H}}_{\mathfrak{K}}, \bar{\pi}_{\mathfrak{K}})$ with $(\bar{\mathscr{H}}^0_{\mathfrak{K}}, \bar{\pi}^0_{\mathfrak{K}})$.

THEOREM 3.1. *If $\{K_i\}_{i=1}^{\infty}$ is a sequence in $\mathscr{T}_1(\mathscr{H})$, the following conditions are equivalent:*

(3.4) *σ extends to an automorphism, also denoted by σ, of the von Neumann algebra $\pi_{\mathfrak{K}}(\overline{F(\mathscr{H})})''$.*

(3.5) *σ is implemented by a unitary on $\bar{\mathscr{H}}_{\mathfrak{K}}$.*

(3.6)
$$\sum_{i=1}^{\infty} (1 - \mathrm{tr}(K_i^{1/2}K_{i+1}^{1/2})) < \infty .$$

In this case the shift on $\bigotimes_{-\infty}^{\infty} (H_i, \Omega_i) = \bar{\mathscr{H}}_{\mathfrak{K}}$ exists and implements σ.

Proof. We first prove the equivalence of (3.6) and (3.4).

Step 1. For a product state, the modification of K_i $(i > 0)$ to K_i' such that $\mathrm{tr}(K_i')^2 = 1$, $K_i' > 0$ and $\sum_i \|K_i - K_i'\|^2_{\text{H-S}} < \infty$ yields the same von Neumann algebras (spatially, the one with K_i is a restriction of the other).

Since

$$\sum_i |\mathrm{tr}(K_i'^{1/2}K_{i+1}'^{1/2}) - \mathrm{tr}(K_i^{1/2}K_{i+1}^{1/2})| \leqslant$$

$$\leqslant \sum_i |\mathrm{tr}(K_i'^{1/2} - K_i^{1/2})K_{i+1}'^{1/2}| + \sum_i |\mathrm{tr}(K_i^{1/2}(K_{i+1}'^{1/2} - K_{i+1}^{1/2}))| \leqslant$$

$$\leqslant \sum_i \{[\mathrm{tr}(K_i'^{1/2} - K_i^{1/2})^2]^{1/2} + [\mathrm{tr}(K_{i+1}'^{1/2} - K_{i+1}^{1/2})^2]^{1/2}\} < \infty .$$

this change does not alter the problem. So we assume $K_i > 0$ for all $i > 0$.

For (3.6), we may restrict our attention to the GNS representation of the state $\psi_{\mathfrak{K}}$ (equivalently the cyclic subspace of $\pi_{\mathfrak{K}}(\overline{F(\mathscr{H})})''$ on the product vector $\bigotimes_{i=-\infty}^{\infty} (K_i^{1/2})$ with $K_i = e_1 \otimes \bar{e}_1$ for $i \leqslant 0$) because the difference with $\pi_{\mathfrak{K}}$ is only for C in $\overline{F(\mathscr{H})} = C \otimes F(\mathscr{H})$ and does not affect the von Neumann algebra.

Step 2. Instead of shifting the algebra, we can shift the state. The problem is then the comparison of two product vectors $\otimes \{K_i^{1/2}\}$ and $\otimes \{L_i^{1/2}\}$ with $L_i = K_{i+1}$ for the same product C^*-algebra $\overline{F(\mathscr{H})}$. Since $K_i = L_i$ for $i < 0$, we can restrict our attention to $i \geqslant 0$. Furthermore we may modify K_0 to K_1 so that $K_0 = L_0$. Thus we have only to compare $\xi = \bigotimes_{i=1}^{\infty} K_i^{1/2}$ and $\eta = \bigotimes_{i=1}^{\infty} L_i^{1/2}$.

Since (3.6) implies that $\bigotimes_{i=1}^{\infty} (\mathscr{H}, K_i^{1/2})$ and $\bigotimes_{i=1}^{\infty} (\mathscr{H}, L_i^{1/2})$ can be identified, (3.6) implies (3.4). We prove the converse. (Note that ξ and η are cyclic by Step 1.)

Step 3. Since the vector at each i is cyclic and separating, the state satisfies the KMS condition for the product of modular automorphism (as C^*-algebra) and hence cyclic and separating (on the weak closure). If ρ and ρ' are faithful states on one von Neumann algebra M, then $\|\rho - \rho'\| < 2$. This is because $\|\rho - \rho'\| \leqslant \|\xi_\rho - \xi_{\rho'}\| \, \|\xi_\rho + \xi_{\rho'}\|$ for vector representative ξ's in the positive cone, then $\xi_\rho, \xi_{\rho'}) > 0$ for faithful ρ and ρ' which implies

$$\|\xi_\rho - \xi_{\rho'}\|^2 \|\xi_\rho + \xi_{\rho'}\|^2 = 4(1 - (\xi_\rho, \xi_{\rho'})) < 4.$$

Step 4. If M_1 is a subalgebra of M, then

$$\|\rho \mid M_1 - \rho' \mid M_1\| \leqslant \|\rho - \rho'\| < 2.$$

We take a finite product as a subalgebra. Then $\xi_N \equiv \bigotimes_{i=1}^{N} K_i^{1/2}$ and $\eta_N \equiv \bigotimes_{i=1}^{N} L_i^{1/2}$ are representative vectors in the positive cone and hence

$$\|\xi_N - \eta_N\|^2 \leqslant \|\rho \mid M_1 - \rho' \mid M_1\| \leqslant \|\rho - \rho'\| < 2.$$

Hence

$$\prod_{i=1}^{N} \operatorname{tr}(K_i^{1/2} L_i^{1/2}) = (\xi_N, \eta_N) \geqslant 1 - (\|\rho - \rho'\|/2) > 0.$$

Since $0 < \operatorname{tr}(K_i^{1/2} L_i^{1/2}) \leqslant 1$, this implies

$$\sum_{i=1}^{\infty} (1 - \operatorname{tr}(K_i^{1/2} L_i^{1/2})) < \infty.$$

The question of the equivalence of (3.4) and (3.5) is whether an isomorphism is spatial. Since we are dealing with an automorphism of a factor, it is always spatial.

REMARK 3.3. If $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$ is as above, then $\bar{\omega}_{\mathfrak{K}}$ is a faithful weight on $\overline{O(\mathscr{H})}$, and if $\mathfrak{N}^{\mathfrak{K}} = \{x \in \overline{O(\mathscr{H})} : \bar{\omega}_{\mathfrak{K}}(x^* x) < \infty\}$, there exists an injective map

$\Lambda^{\mathfrak{K}}$ from $\mathfrak{N}^{\mathfrak{K}}$ into a Hilbert space $\mathscr{H}^{\mathfrak{K}}$ and a representation $\bar{\pi}^{\mathfrak{K}}$ of $\overline{O(\mathscr{H})}$ on such that

$$\langle \Lambda^{\mathfrak{K}}(x), \Lambda^{\mathfrak{K}}(y)\rangle = \bar{\omega}_{\mathfrak{K}}(y^*x), \quad x, y \in \mathfrak{N}^{\mathfrak{K}} \tag{3.7}$$

$$\bar{\pi}^{\mathfrak{K}}(x)\Lambda^{\mathfrak{K}}(y) = \Lambda^{\mathfrak{K}}(xy), \quad x \in \overline{O(\mathscr{H})}, \ y \in \mathfrak{N}^{\mathfrak{K}}, \tag{3.8}$$

$$\{\Lambda^{\mathfrak{K}}(y) : y \in \mathfrak{N}^{\mathfrak{K}}\}^- = \mathscr{H}^{\mathfrak{K}}. \tag{3.9}$$

The triple $(\mathscr{H}^{\mathfrak{K}}, \bar{\pi}^{\mathfrak{K}}, \Lambda^{\mathfrak{K}})$ can be identified as follows:

Under the identification (2.4), a dense subalgebra of the algebraic tensor product $C_0 \otimes O_0(\mathscr{H})$ embeds into $\overline{O_0(\mathscr{H})}$, where $O_0(\mathscr{H})$ is the $*$-algebra generated by $\{O(h) = h \in \mathscr{H}\}$, and $\overline{O_0(\mathscr{H})}$ is the $*$-subalgebra of $\overline{O(\mathscr{H})} = C^*(\overline{F(\mathscr{H})}, \mathbf{Z})$ generated by $\overline{F_0(\mathscr{H})}$ and the shift. Hence as for $\bar{\rho}_{\mathfrak{K}}$, we see that $\Lambda^{\mathfrak{K}}(\overline{O_0(\mathscr{H})})$ is dense in $\mathscr{H}^{\mathfrak{K}}$. If we think of $\overline{O_0(\mathscr{H})}$ as functions from $\mathbf{Z}$ into $\overline{F_0(\mathscr{H})}$ with finite support and twisted convolution, define a map $\Lambda_0^{\mathfrak{K}}$ from $\overline{O_0(\mathscr{H})}$ into $\ell^2(\mathbf{Z}, \mathscr{H}_{\mathfrak{K}})$ by

$$\Lambda_0^{\mathfrak{K}}(f) = \bigoplus_{n=-\infty}^{\infty} \Lambda_{\mathfrak{K}}(f(n)), \quad f \in \overline{O_0(\mathscr{H})}.$$

Then $\langle \Lambda_0^{\mathfrak{K}}(f), \Lambda_0^{\mathfrak{K}}(g)\rangle = \bar{\omega}_{\mathfrak{K}}(g^*f) = \langle \Lambda^{\mathfrak{K}}(f), \Lambda^{\mathfrak{K}}(g)\rangle$. Hence we can identify $\ell^2(\mathbf{Z}, \mathscr{H}_{\mathfrak{K}}) = [\Lambda_0^{\mathfrak{K}}(O_0(\mathscr{H}))]^-$ with $\mathscr{H}^{\mathfrak{K}} = [\Lambda^{\mathfrak{K}}(\overline{O_0(\mathscr{H})})]^-$, and then $\bar{\pi}^{\mathfrak{K}}$ is the representation of the crossed product $C^*(\overline{F(\mathscr{H})}, \mathbf{Z})$ induced from the representation $\pi_{\mathfrak{K}}$ of $\overline{F(\mathscr{H})}$. Then as in the proof of [9, Theorem 3.1], if σ extends to an automorphism of $\pi_{\mathfrak{K}}(\overline{F(\mathscr{H})})'' = \mathfrak{N}_{\mathfrak{K}}$ we can identify $\pi^{\mathfrak{K}}(O(H))'' = \mathfrak{M}_{\mathfrak{K}}$ with the crossed product $W^*(\mathfrak{N}_{\mathfrak{K}}, \mathbf{Z})$.

REMARK 3.4. Similarly if n is finite, and $\{K_i\}_{i=1}^{\infty}$ a commutative family in $\mathscr{T}_1(\mathscr{H})$, then by [10], the associated "product" measure $\mu_{\mathfrak{K}}$ on X_n is quasi-invariant under G_n if and only if

$$K_i > 0 \quad i = 1,2, \ldots \tag{3.10}$$

$$\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} K_{i+1}^{1/2}) < \infty. \tag{3.11}$$

In this case $\mathfrak{M}_{\mathfrak{K}} = \bar{\pi}^{\mathfrak{K}}(O(\mathscr{H}))''$ is isomorphic to the W^*-crossed product $W^*(X_n,$ $\mu_{\mathfrak{K}}, G_n)$. It is clear that $(X_n, \mu_{\mathfrak{K}})$ is G_n-ergodic since it is already ergodic. If $\mu_{\mathfrak{K}}$ is not discrete, then for each g in G_n, $g \neq 0$, $\{x : gx = x\}$ is countable, hence of $\mu_{\mathfrak{K}}$-measure zero. Thus $(X_n, \mu_{\mathfrak{K}}, G_n)$ is free. In this case $W^*(X_n, \mu_{\mathfrak{K}}, G_n)$ is a factor [12].

More generally, in the case where the $\{K_i\}$ do not necessarily form a commutative family:

THEOREM 3.5. *Suppose* $2 \leqslant n < \infty$, *and* $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$ *a sequence in* $\mathcal{T}_1(\mathcal{H})$. *Then the following conditions are equivalent:*

(3.12) $\omega_{\mathfrak{K}}$ *is not factorial*

(3.13) $\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} K_{i+s}^{1/2}) < \infty$ *for some* $s > 0$, *and* $\rho_{\mathfrak{K}}$ *is type* I.

Proof. We first show that if

$$\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} K_{i+s}^{1/2}) < \infty \tag{3.14}$$

$$\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} K_{i+r}^{1/2}) = \infty \quad \text{for } 0 < r < s \tag{3.15}$$

for some $s > 0$ then

$$\operatorname{center} \bar{\pi}^{\mathfrak{K}}(\overline{O(\mathcal{H})})'' \simeq \operatorname{center} W^*(\bar{\pi}_{\mathfrak{K}}(\overline{F(\mathcal{H})}), \sigma^s) \tag{3.16}$$

and if

$$\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} K_{i+r}^{1/2}) = \infty \quad \text{for all } r > 0. \tag{3.17}$$

Then $\bar{\pi}^{\mathfrak{K}}(\overline{O(\mathcal{H})})''$ is a factor.

Write $\mathcal{H}^{\mathfrak{K}} = \ell^2(\mathbf{Z}, \mathcal{H}_{\mathfrak{K}}) = \bigoplus_{i \in \mathbf{Z}} \mathcal{H}_i$, where $\mathcal{H}_i = \mathcal{H}_{\mathfrak{K}}$, and represent an operator T on $\bigoplus_i \mathcal{H}_i$ by a matrix $[T_{ij}]$, where $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$. The representation $\bar{\pi}^{\mathfrak{K}}$ of $\overline{O(\mathcal{H})}$ is the representation obtained from the following covariant representation (θ, λ) of $(\overline{F(\mathcal{H})}, \mathbf{Z}, \sigma)$:

$$\theta(x) = [\bar{\pi}_{\mathfrak{K}} \sigma^{-i}(x)]_{i \in \mathbf{Z}},$$

a diagonal matrix

$$(\lambda(m) f)(n) = f(n - m)$$

for $x \in \overline{F(\mathcal{H})}$, $f \in \ell^2(\mathbf{Z}, \mathcal{H}_{\mathfrak{K}})$, $m, n \in \mathbf{Z}$.

Let p_0 denote the projection of $\bigoplus_{i\in Z} \mathscr{H}_i$ on $\bigoplus_{i\in Z} \mathscr{H}_{is}$. Then if (3.14) and (3.1 hold

$$W^*(\bar{\pi}^{\mathfrak{K}}(\overline{F(\mathscr{H})})p_0)\ \lambda(s)p_0, \simeq W^*(\bar{\pi}_{\mathfrak{K}}(\overline{F(\mathscr{H})}'', \sigma^s)$$

where the left hand side is the von Neumann algebra generated by $\{\bar{\pi}^{\mathfrak{K}}(x)p_0, \lambda(s)$ $:x \in \overline{F(\mathscr{H})}\}$ and the right hand side is the W^*-crossed product which exists (3.14). Let $T = [T_{ij}] \in \bar{\pi}^{\mathfrak{K}}(\overline{O(\mathscr{H})})'$. Then since T commutes with the shift λ,

$$T_{ij} = S(i-j)$$

for some map $S\colon Z \to \mathscr{B}(\mathscr{H}_{\mathfrak{K}})$. Moreover, since T commutes with $\theta(x), x \in \overline{F(}$ we have:

$$\bar{\pi}_{\mathfrak{K}}(\sigma^{-i}(x))S(i-j) = S(i-j)\bar{\pi}_{\mathfrak{K}}(\sigma^{-j}(x))$$

for $x \in \overline{F(\mathscr{H})}$, $i, j \in Z$. By (3.15) and Theorem 3.1, $\bar{\pi}_{\mathfrak{K}}\sigma^{-i}$ and $\bar{\pi}_{\mathfrak{K}}\sigma^{-j}$ are disj representations if $i-j$ is not divisible by s. Hence $S(r) = 0$ if s does not divid Hence we see that the map $T \to Tp_0$ takes

$$\bar{\pi}^{\mathfrak{K}}(\overline{O(\mathscr{H})})' \text{ onto } W^*(\bar{\pi}_{\mathfrak{K}}(\overline{F(\mathscr{H})})'', \sigma^s)'$$

and moreover it is easily seen that this map takes

$$\bar{\pi}^{\mathfrak{K}}(\overline{O(\mathscr{H})})' \cap [\bar{\pi}^{\mathfrak{K}}(\overline{O(\mathscr{H})})']'$$

onto

$$W^*(\bar{\pi}'_{\mathfrak{K}}(F(\mathscr{H}))'', \sigma^s)' \cap [W^*(\bar{\pi}_{\mathfrak{K}}(\overline{F(\mathscr{H})})'', \sigma^s)']'.$$

The remaining claim concerning (3.17) is now clear. Theorem 3.4 is now a conse quence of the following lemma:

Lemma 3.6. *If* $\sum_{i=1}^{\infty}(1 - \operatorname{tr} K_i^{1/2}K_{i+1}^{1/2}) < \infty$, *and* $\rho_{\mathfrak{K}}$ *is not type* I, *th* $W^*(\bar{\pi}_{\mathfrak{K}}(\overline{F(\mathscr{H})})'', \sigma)$ *is a factor.*

Proof. By Connes [5] it is enough to show that every non-trivial power of th shift on $\bar{\pi}_{\mathfrak{K}}(\overline{F(\mathscr{H})})''$ is outer. By regrouping it is enough to consider the case of th shift σ itself. Thus suppose σ is inner, implemented by a unitary W in $\bigoplus_{i\in Z}(M_n, \Omega_i$

Regarding M_n as a Hilbert space (containing $\Omega_i = K_i^{1/2}$), on which M_n acts, let

$$A_i = \begin{cases} C^*(M_n,\ M_n') = B(M_n) & i \text{ even} \\ 1 & i \text{ odd.} \end{cases}$$

Then $A = \bigotimes_{i \in \mathbf{Z}} (A_i, \Omega_i)$ is type I. Let $\beta = \mathrm{Ad}(w)|A$. Then

$$A \simeq \beta A = \bigotimes_{i \in \mathbf{Z}} (B_i,\ \Omega_i)$$

where

$$B_i = \begin{cases} M_n & i \text{ odd} \\ M_n' & i \text{ even.} \end{cases}$$

Then $A \simeq [\bigotimes_i (M_n,\ \Omega_{2i+1})] \otimes [\bigotimes_i (M_n',\ \Omega_{2i})]$.

Hence $\bigotimes_i (M_n, \Omega_{2i+1})$ is type I, and similarly, so is $\bigotimes_i (M_n, \Omega_{2i})$. This contradicts $\rho_{\mathfrak{K}}$ not being type I, and the lemma follows.

4. QUASI-FREE AUTOMORPHISMS

If u is a unitary on $\mathscr{H}$, we consider the question of implementability of $O(u)$ on $O(\mathscr{H})$ in various quasi-free representations.

If $\mathfrak{K} = \{K_i\}$ is a sequence of operators in $\mathscr{T}_1(\mathscr{H})$, we let $(\pi^{\mathfrak{K}},\ \mathscr{H}^{\mathfrak{K}},\ \Omega^{\mathfrak{K}})$ denote the GNS triplet for $\omega_{\mathfrak{K}}$. The GNS triplet $(\pi_{\mathfrak{K}}, \mathscr{H}_{\mathfrak{K}}, \Omega_{\mathfrak{K}})$ of $\rho_{\mathfrak{K}}$ can then be identified with the representation $\pi_{\mathfrak{K}}(\cdot) = \pi(\cdot)\,|\,\mathscr{H}_{\mathfrak{K}}$, on $\mathscr{H}_{\mathfrak{K}} = [F(\mathscr{H})\Omega^{\mathfrak{K}}]^-$, with cyclic vector $\Omega_{\mathfrak{K}} = \Omega^{\mathfrak{K}}$. Since $\omega_{\mathfrak{K}}$ is gauge invariant, there is a strongly continuous unitary group $\{F_t : t \in \mathbf{T}\}$ on $\mathscr{H}^{\mathfrak{K}}$ such that

$$\pi^{\mathfrak{K}}\alpha_t(x) = F_t \pi^{\mathfrak{K}}(x) F_t^*; \quad x \in O(\mathscr{H}), \tag{4.1}$$

$$F_t \Omega_{\mathfrak{K}} = \Omega_{\mathfrak{K}}. \tag{4.2}$$

We let $\tilde{\alpha}_t = \mathrm{Ad}\, F_t$ denote the induced action on $\mathfrak{M}_{\mathfrak{K}} = \pi^{\mathfrak{K}}(O(\mathscr{H}))''$, and let $\mathfrak{N}_{\mathfrak{K}} = \pi_{\mathfrak{K}}(F(\mathscr{H}))''$.

We first prove a positive result.

Proposition 4.1. *Let $2 \leqslant n < \infty$, $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$ a sequence of operators in $\mathscr{T}_1(\mathscr{H})$ and u a unitary on $\mathscr{H}$.*

(a) *$F(u)$ is implementable in $\rho_{\mathfrak{K}}$ if and only if* $\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} u K_i^{1/2} u^*) < \infty$. *If $F(u)$ is implementable in $\rho_{\mathfrak{K}}$ then $O(u)$ is implementable in $\omega_{\mathfrak{K}}$.*

(b) *$F(u)$ is weakly inner in $\rho_{\mathfrak{K}}$ if and only if*

$$\sum_{i=1}^{\infty} (1 - |\operatorname{tr} u K_i|) < \infty.$$

When this, and the equivalent conditions of Theorem 3.1 *hold, then $O(u)$ is weakly inner in $\omega_{\mathfrak{K}}$.*

Proof. (a) $F(u)$ is implementable if and only if $\bigotimes_{i=1}^{\infty} \rho_{K_i}$ and $\bigotimes_{i=1}^{\infty} \rho_{uK_iu^*}$ are unitarily equivalent, which is the case if and only if

$$\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} u K_i^{1/2} u^*) < \infty.$$

Suppose that $F(u)$ is implementable by a unitary U_0 on $\mathscr{H}_{\mathfrak{K}}$. Then $U_0\Omega_{\mathfrak{K}} \in \mathscr{H}$ shows that $F_t U_0 \Omega_{\mathfrak{K}} = U_0 \Omega_{\mathfrak{K}}$ for all $t \in \mathrm{T}$. Hence for $e_1, \ldots, e_l, f_1, \ldots, f_r$, $g_1, \ldots, g_s, h_1, \ldots, h_v \in \mathscr{H}$ we have

$$\langle O(ue_1) \ldots O(ue_l) O(uf_r)^* \ldots O(uf_1)^* U_0 \Omega_{\mathfrak{K}},$$

$$O(ug_1) \ldots O(ug_s)\, O(uh_v)^* \ldots O(uh_1)^* U_0 \Omega_{\mathfrak{K}} \rangle =$$

$$= \langle U_0^* O(uh_1) \ldots O(uh_v) O(ug_s)^* \ldots O(ug_1)^* O(ue_1) \ldots$$

$$\ldots O(ue_l)\, O(uf_r)^* \ldots O(uf_1)^* U_0 \Omega_{\mathfrak{K}}, \Omega_{\mathfrak{K}} \rangle =$$

$$= \langle U_0^* F_t O(Uh_1) \ldots O(Uf_1)^* F_t^* U_0 \Omega_{\mathfrak{K}}, \Omega_{\mathfrak{K}} \rangle =$$

$$= \langle U_0^* O(Uh_1) \ldots O(Uf_1)^* U_0 \Omega_{\mathfrak{K}}, \Omega_{\mathfrak{K}} \rangle\, \delta_{l+t,\, r+s} =$$

$$= \langle O(h_1) \ldots O(f_1)^* \Omega_{\mathfrak{K}}, \Omega_{\mathfrak{K}} \rangle\, \delta_{l+t,\, r+s} = \quad (\text{as } F(u) = \operatorname{Ad} U_0)$$

$$= \langle O(e_1) \ldots O(e_l) O(f_r)^* \ldots O(f_1)^* \Omega_{\mathfrak{K}}, O(g_1) \ldots O(g_s) O(h_v)^* \ldots O(h_1)^* \Omega_{\mathfrak{K}} \rangle.$$

Hence there exists a well defined unitary U on $\mathscr{H}^{\mathfrak{K}}$ such that $U(x\Omega_{\mathfrak{K}}) = O(u)(x) U_0 \Omega_{\mathfrak{K}}$ for $x \in O(\mathscr{H})$. In which case $O(u)(x) = UxU^*$ for $x \in O(\mathscr{H})$.

(b) The argument of Connes [5, 1.3.8] establishes the first part. Suppose that U_0 implements $F(u)$ in $\pi_{\mathfrak{K}}$, and $U_0 \in \pi_{\mathfrak{K}}(F(\mathscr{H}))''$. Then the only extension of U_0 which implements $F(u)$ is the one defined in (a), and we have to show it lies in $\pi^{\mathfrak{K}}(O(\mathscr{H}))''$.

Regard $\pi^{\mathfrak{K}}$ as the representation

$$a \to \bar{\pi}^{\mathfrak{K}}(faf), \quad a \in O(\mathscr{H}) \text{ on } \bar{\pi}^{\mathfrak{K}}(f)\overline{\mathscr{H}}$$

where (§ 2) $f = \bigotimes_{-\infty}^{\infty} f_i$ is the identity of $F(\mathscr{H})$. Moreover $\bar{\pi}^{\mathfrak{K}}$ is the representation on $\overline{\mathscr{H}}^{\mathfrak{K}} = \ell^2(\mathbf{Z}, \overline{\mathscr{H}}_{\mathfrak{K}})$ induced from $\bar{\pi}_{\mathfrak{K}}$. Thus $\bar{\pi}^{\mathfrak{K}}$ is obtained from the covariant representation of $(\overline{F(\mathscr{H})}, \mathbf{Z}, \sigma)$ given by:

$$(\lambda^m \varphi)(s) = \varphi(s - m)$$

$$(\theta(a)\,\varphi)(s) = \pi_{\mathfrak{K}}(\sigma^{-s}(a))\varphi(s),$$

for $a \in \overline{F(\mathscr{H})}$, $\varphi \in \ell^2(\mathbf{Z}, \overline{\mathscr{H}}_{\mathfrak{K}})$.

Since σ extends to $\bar{\pi}_{\mathfrak{K}}(\overline{F(\mathscr{H})})''$, we can define $\theta(U_0) \in \theta(\overline{F(\mathscr{H})})'' \subseteq \bar{\pi}^{\mathfrak{K}}(O(\mathscr{H}))''$. According to Connes [5, 1.3.8] U_0 can be taken as $\left(\bigotimes_{-\infty}^{0} f_i\right) \otimes \left(\bigotimes_{1}^{\infty} u\right)$ on $\overline{\mathscr{H}}_{\mathfrak{K}}$. Thus

$$\theta(U_0) = \text{strong} \lim_{k \to \infty} \bar{\pi}^{\mathfrak{K}}\left(\bigotimes_{-\infty}^{0} f_i \otimes \bigotimes_{1}^{k} u \bigotimes_{k+1}^{\infty} 1\right).$$

According to Cuntz [6], under the identification (2.4),

$$\bar{\pi}^{\mathfrak{K}} O(h) = \bar{\pi}^{\mathfrak{K}}\left(\bigotimes_{-\infty}^{0} f_i \otimes [h \otimes \bar{e}_1] \otimes \bigotimes_{2}^{\infty} 1\right) \lambda_1 \bar{\pi}^{\mathfrak{K}}\left(\bigotimes_{-\infty}^{0} f_i \otimes \bigotimes_{1}^{\infty} 1\right).$$

Then

$$\theta(U_0)\bar{\pi}^{\mathfrak{K}}(O(h))\theta(U_0)^* =$$

$$= \text{st}\lim_{k \to \infty} \bar{\pi}^{\mathfrak{K}}\left[\left(\bigotimes_{-\infty}^{0} f_i \otimes \bigotimes_{1}^{k} u \otimes \bigotimes_{k+1}^{\infty} 1\right)\left(\bigotimes_{-\infty}^{0} e \otimes [h \otimes \bar{e}_1] \otimes \bigotimes_{2}^{\infty} 1\right)\right] \times$$

$$\times\, \lambda_1 \bar{\pi}^{\mathfrak{K}}\left(\bigotimes_{-\infty}^{0} f_i \otimes \bigotimes_{1}^{k} u^* \bigotimes_{k+1}^{\infty} 1\right) =$$

$$= \text{st}\lim_{n \to \infty} \bar{\pi}^{\mathfrak{K}}\left\{\bigotimes_{-\infty}^{0} f_i \otimes [uh \otimes \bar{e}_1] \otimes \bigotimes_{2}^{k} u \otimes \bigotimes_{k+1}^{\infty} 1\right\} \times$$

$$\times\ \lambda_1 \bar{\pi}^{\mathfrak{K}}\left(\bigotimes_{-\infty}^{0} f_i \otimes \bigotimes_{1}^{k} u^* \otimes \bigotimes_{k+1}^{\infty} 1 \right) =$$

$$= \operatorname*{st\,lim}_{n\to\infty} \bar{\pi}^{\mathfrak{K}}\left\{ \bigotimes_{-\infty}^{0} f_i \otimes [uh \otimes \bar{e}_i] \otimes \bigotimes_{2}^{k} u \otimes \bigotimes_{k+1}^{\infty} 1 \right\} \times$$

$$\times\ \bar{\pi}^{\mathfrak{K}}\left(\bigotimes_{-\infty}^{1} f_{i-1} \otimes \bigotimes_{2}^{k+1} u^* \otimes \bigotimes_{k+2}^{\infty} 1 \right) \lambda_1 =$$

$$= \bar{\pi}^{\mathfrak{K}}\left(\bigotimes_{-\infty}^{0} f_i \otimes [uh \otimes \bar{e}_1] \otimes \bigotimes_{2}^{\infty} 1 \right) \lambda_1 =$$

$$= \bar{\pi}^{\mathfrak{K}}(O(uh)).$$

By reducing with $\bar{\pi}^{\mathfrak{K}}(f)$, we see that $O(u)$ is implemented by $\bar{\pi}^{\mathfrak{K}}(f)O(U_0)\bar{\pi}^{\mathfrak{K}}(f) \in \bar{\pi}^{\mathfrak{K}}(O(\mathscr{H}))''$ in $\omega_{\mathfrak{K}}$.

REMARK 4.2. Note that if $\mathfrak{K}$ is a commutative family, say $K_i = \sum_{j=1}^{n} \varkappa_{ij} e_j \otimes \bar{e}_j$, and u is the unitary $[u_{kl}]$, then $\operatorname{tr} K_i^{1/2} u K_i^{1/2} u^* = \sum_{k,l} |u_{kl}|^2 \varkappa_{ik}^{1/2} \varkappa_{il}^{1/2}$. There exists a unitary u with $|u_{kl}|^2 = 1/n$. If the conditions of the above proposition hold for this particular u, then

$$\sum_{i=1}^{\infty} (1 - [\operatorname{tr}(K_i^{1/2})]^2)/n < \infty.$$

In which case

$$\sum_{i=1}^{\infty} (1 - (\operatorname{tr} K_i^{1/2})/n^2)) < \infty,$$

and so $\rho_{\mathfrak{K}}$ is equivalent to the trace on $F(\mathscr{H})$, and hence all of $F(U(\mathscr{H}))$ is implementable in $\rho_{\mathfrak{K}}$.

THEOREM 4.3. *Let* $2 \leqslant n < \infty$, *and* $\mathfrak{K} = \{K_i\}_{i=1}^{\infty}$ *a sequence in* $\mathscr{T}_1(\mathscr{H})$ *with*

(4.3) $$K_i > 0 \quad \textit{for all } i$$

(4.4) $\rho_{\mathfrak{K}}$ *is not type* I.

(4.5) $$\sum_{i=1}^{\infty} (1 - \operatorname{tr} K_i^{1/2} K_{i+1}^{1/2}) < \infty.$$

Then

(4.6) $\{\alpha_t : t \in \mathbf{T} \setminus 1\}$ *are not weakly inner in* $\omega_{\mathfrak{N}}$.

(4.7) *If* $\{O(v) : v \in \mathrm{SU}(n)\}$ *is implementable in* $\omega_{\mathfrak{N}}$, *and if* $O(u)$, *for* $u \in \mathrm{SU}(n)$, *is weakly inner in* $\omega_{\mathfrak{N}}$ *then* $F(u)$ *is weakly inner in* $\rho_{\mathfrak{N}}$.

Proof. (4.6) For this it is convenient to work with the weight $\bar{\omega}_{\mathfrak{N}} = \mathrm{tr} \otimes \omega_{\mathfrak{N}}$ on $\mathscr{C} \otimes O(\mathscr{H})$. Then with $\bar{\mathfrak{M}} = \bar{\pi}^{\mathfrak{N}}(O(\mathscr{H}))''$, $\mathfrak{M} = \pi^{\mathfrak{N}}(O(\mathscr{H}))''$ and $\bar{\mathfrak{M}} = \mathscr{C}'' \otimes M$ suppose for some $s \in \mathbf{T}$, α_s is implemented in $\omega_{\mathfrak{N}}$ by a unitary in $\mathfrak{M}$. Then $\hat{\sigma}_s = 1 \otimes \alpha_s$ is implemented by a unitary in $\bar{\mathfrak{M}}$. Now by Remark 3.3, $\bar{\mathfrak{M}} \cong W^*(\bar{\mathfrak{N}}, \mathbf{Z})$ if $\bar{\mathfrak{N}} = \bar{\pi}_{\mathfrak{N}}(\overline{F}(\mathscr{H}))''$, and under this identification, the dual action $\hat{\sigma}$ of σ on $\overline{O(\mathscr{H})}$ extends to $W^*(\bar{\mathfrak{N}}, \mathbf{Z})$ and corresponds to the dual action of the extension of σ to $\bar{\mathfrak{N}}$. Let $W \in \bar{\mathfrak{M}}$ be the canonical unitary implementing the shift σ on $\bar{\mathfrak{N}} = \bar{\pi}_{\mathfrak{N}}(F)''$. Then the dual action $\hat{\sigma}_t$ is given by

$$\hat{\sigma}_t \,|\, \bar{\mathfrak{N}} = \mathrm{id}, \quad \hat{\sigma}_t(W) = tW, \quad t \in \mathbf{T}.$$

Now suppose $\hat{\sigma}_s = \mathrm{Ad}(V) \,|\, \bar{\mathfrak{M}}$ for some V in $\bar{\mathfrak{M}}$, and some s in $\mathbf{T}$. Then $\hat{\sigma}_s\hat{\sigma}_t = \hat{\sigma}_t\hat{\sigma}_s$, $t \in \mathbf{T}$, shows that $\hat{\sigma}_t(V)V^* \in \bar{\mathfrak{M}} \cap \bar{\mathfrak{M}}'$. But $\bar{\mathfrak{M}}$ is a factor by Theorem 3.5 and assumption (4.4). Hence $\hat{\sigma}_t(V) = t^m V$ for some $m \in \mathbf{Z}$. Then $W^m V^* \in \bar{\mathfrak{M}}^{\mathbf{T}} = \bar{\mathfrak{N}}$ and implements σ^m. But by the proof of Lemma 3.6, no non-trivial power of the shift on $\bar{\mathfrak{N}}$ is inner, hence $m=0$ and $V \in \bar{\mathfrak{N}}$. Then $\mathrm{id} = F(s) = \mathrm{Ad}\, V \,|\, \bar{\mathfrak{N}}$ shows that $V \in \bar{\mathfrak{N}} \cap \bar{\mathfrak{N}}'$. But $\bar{\mathfrak{N}}$ is a factor, hence $\hat{\sigma}_s = \mathrm{Ad}(V) \,|\, \bar{\mathfrak{N}} = \mathrm{id}$ and so $s = 1$.

(4.7) Since $\mathrm{SU}(n)$ is implementable in $\omega_{\mathfrak{N}}$, the subgroup of $\mathrm{SU}(n)$ which is inner in $\omega_{\mathfrak{N}}$ is normal. Hence if there exists $u \in \mathrm{SU}(n) \setminus (\mathbf{T} \cap \mathrm{SU}(n))$ such that $O(u)$ is inner in $\omega_{\mathfrak{N}}$, then all of $\mathrm{SU}(n)$ is inner in $\omega_{\mathfrak{N}}$. Now $\mathrm{SU}(n)$ is the union of conjugates of $\mathbf{T}^{n-1}$, so let $v : \mathbf{T}^{n-1} \to \mathrm{SU}(n)$ be a homomorphism. Then $O(v(s))$, $s \in \mathbf{T}^{n-1}$ is implemented by $V(s) \in \mathfrak{M} = \pi^{\mathfrak{N}}(O(\mathscr{H}))''$ say. By §3, $\mathfrak{M}$ is a factor and so

$$V(s)V(s') = d(s, s')\, V(s + s') \qquad s, s' \in \mathbf{T}^{n-1}$$

for some map $d : (\mathbf{T}^{n-1})^2 \to \mathbf{T}$. Moreover $\alpha_t O(v(s)) = O(v(s))\alpha_t$ $t \in \mathbf{T}$, $s \in \mathbf{T}^{n-1}$, and the implementability of α_t (4.1), by F_t, and again the factoriality of $\mathfrak{M}$ shows that

$$V(s)F_t = c(t, s)F_t V(s) \qquad t \in \mathbf{T},\ s \in \mathbf{T}^{n-1}$$

for some map $c : \mathbf{T} \otimes \mathbf{T}^{n-1} \to \mathbf{T}$.

Then $F_t V(s) F_{t'} V(s') = c(t', s)\, d(s, s') F_{t+t'} V(s + s')$ for $t, t' \in \mathbf{T}$, $s, s' \in \mathbf{T}^{n-1}$.

Let $\sigma((t, s), (t', s')) = c(t', s)d(s, s')$, so that $\sigma = (\mathbf{T} \times \mathbf{T}^{n-1})^* \to \mathbf{T}$ is a 2-cocycle. Now d is a cocycle. Hence the cocycle conditions for σ imply $c(t't'', s) \cdot c(t'', s') = c(t'', ss')\, c(t', s)$. Using $c(1, s) = c(t, 1) = 1$, we see that c is a bicharacter (set $s' = 1$ or $t' = 1$). Thus $y \to c(\cdot, y)$ is a homomorphism from $\mathbf{T} \times \mathbf{T}^{n-1}$ into $(\mathbf{T} \times \mathbf{T}^{n-1})^* = \mathbf{Z}^n$. But $\mathbf{T} \times \mathbf{T}^{n-1}$ is divisible, and there are no non-zero divisible subgroups of $\mathbf{Z}^n$. Hence $c \equiv 1$, and so $F_t V(s)$ commutes with $F_{t'} V(s')$. In particular $F_t V(s) = V(s) F_t$. Then $V(s)$ leaves $[\pi_{\mathcal{K}}(F(\mathcal{H}))\Omega]^- = \mathcal{H}_{\mathcal{K}}$ invariant, and $V^0(s) = V(s)|\mathcal{H}_{\mathcal{K}}$ implements $F(v(s))$. Now $V(s) \in (\pi^{\mathcal{K}}(O(\mathcal{H})'')^{\mathbf{T}}$, and it will follow that $F(v(s))$ is weakly inner if $[\pi^{\mathcal{K}}(O(\mathcal{H}))'']^{\mathbf{T}} = \pi^{\mathcal{K}}(F(\mathcal{H})'')$, since then $V^0(s) \in \pi_{\mathcal{K}}(F(\mathcal{H}))''$.

Thus to complete the proof, note that $\overline{\mathfrak{M}}^{\mathbf{T}} = \overline{\mathfrak{N}}$ was shown in the proof of (a). But $\overline{\mathfrak{M}} = \mathscr{C}'' \otimes \mathfrak{M}$, $\overline{\mathfrak{N}} = \mathscr{C}'' \otimes \mathfrak{N}$, and $\overline{\mathfrak{M}}^{\mathbf{T}} = \mathscr{C}'' \otimes \mathfrak{M}^{\mathbf{T}}$. Hence $\mathfrak{M}^{\mathbf{T}} = \mathfrak{N}$ as desired, since $\pi_{\mathcal{K}}(F(\mathcal{H}))'' \simeq \pi^{\mathcal{K}}(F(\mathcal{H}))''$. Alternatively, one can see that $[\pi^{\mathcal{K}}(O(\mathcal{H}))'']^{\mathbf{T}} = \pi^{\mathcal{K}}(F(\mathcal{H}))''$ is a consequence of faithfulness of $\rho_{\mathcal{K}}$. The states $\mathfrak{M}_*^0 = \{\langle \cdot\, \Omega_{\mathcal{K}}, y'\Omega_{\mathcal{K}}\rangle : y' \in \mathfrak{M}'\}$ are dense in $\mathfrak{M}_*$ because Ω is separating. But

$$\left\langle \int \hat{\alpha}_t(\cdot)\mathrm{d}t\,\Omega_{\mathcal{K}},\; y'\Omega_{\mathcal{K}} \right\rangle = \left\langle (\cdot)\Omega_{\mathcal{K}},\; \int F_t^* y' \Omega_{\mathcal{K}} \mathrm{d}t \right\rangle.$$

Hence with $\rho = \int \hat{\alpha}(\cdot)\mathrm{d}t$ as the projection of $\mathfrak{M}$ onto $\mathfrak{M}^{\mathbf{T}}$, we see that $P^*\mathfrak{M}_*^0 \subseteq \mathfrak{M}_*^0$. Hence $P^*\mathfrak{M}_* \subseteq \mathfrak{M}_*$ and so P is normal. Hence

$$\mathfrak{M}^{\mathbf{T}} = P\,[\pi^{\mathcal{K}}(O(\mathcal{H}))]'' = [P(\pi^{\mathcal{K}}(O(\mathcal{H}))]'' = [\pi^{\mathcal{K}}(F(\mathcal{H}))]''.$$

An immediate corollary of Theorem 4.3 is that $\{\alpha_t : t \in \mathbf{T}\setminus\{1\}\}$ and $\{O(u) : u \in \mathrm{SU}(n)\setminus\{1\}\}$ are not weakly inner on O_n in the canonical state $\omega_{1/n}$, if $n < \infty$. In [1, 7, 9] it was shown that $\{O(u) : u \in U(n)\setminus\{1\}\}$ is outer on O_n, provided $n < \infty$. We now improve on these results to show:

THEOREM 4.5. (a) *If* $2 \leqslant n < \infty$, $\{O(u) : u \in U(\mathcal{H})\setminus\{1\}\}$ *is not weakly inner in* $\omega_{1/n}$.

(b) *If* $n = \infty$, *then* $\{O(u) : u \in U(\mathcal{H})\setminus\{1\}\}$ *is outer on* $O(\mathcal{H})$, *but weakly inner in* ω_0.

Proof. (a) We suppose that $(\pi, \mathcal{H}, \Omega)$ is the GNS triplet for the canonical state $\omega_{1/n}$ on $O(\mathcal{H})$, and $\{F_t : t \in \mathbf{T}\}$ implements the gauge group α, with $F_t\Omega = \Omega$, $t \in \mathbf{T}$. Suppose $u \in O(\mathcal{H})$ and $\pi(O(u)(x)) = U\pi(x)U^*$, $x \in O(\mathcal{H})$ for some $U \in \pi(O(\mathcal{H}))''$. Then $\alpha_t O(u) = O(u)\alpha_t$ shows that $F_t U F_t^* U^* \in \pi(O(\mathcal{H}))'' \cap \pi(O(\mathcal{H}))'$, which is trivial because the canonical state is a factor ([9] and Remark 3.5). Hence $F_t U F_t^* = t^m U$ for some $m \in \mathbf{Z}$, all $t \in \mathbf{T}$. By considering $O(u^*)$ if necessary, we may assume

$m \geqslant 0$. Let $S = O(f)$ for some $f \in \mathscr{H}$, $\|f\| = 1$. Then $A = U\pi(S^*)^m \in M^{\tilde{\alpha}}$ if $\tilde{\alpha}_t = \operatorname{Ad} F_t$ on $\mathfrak{M} = \pi(O(\mathscr{H}))''$. Then $A^*A = \pi(S^m S^{m*})$, $AA^* = 1$, and so $\pi(S^m S^{m*})$ is equivalent to 1 in $\mathfrak{M}^{\mathrm{T}} = \pi(F(\mathscr{H}))''$ (using remarks in the proof of Theorem 4.3). But $\langle(\cdot)\,\Omega, \Omega\rangle$ is a trace on $\pi(F(\mathscr{H}))''$. Thus $n^{-m} = \langle \pi(S^m S^{m*})\Omega, \Omega\rangle = \langle 1\Omega, \Omega\rangle = 1$. Hence $m = 0$, and $U \in \pi(F(\mathscr{H}))''$. Thus U is reduced by $[\pi(F(\mathscr{H}))\Omega]^- = \mathscr{H}^0$, and $U_0 = U|\mathscr{H}^0$ implements $F(u)$ in $\rho_{1/n}$. Hence $F(u)$ is weakly inner in $\rho_{1/n}$. Then Proposition 4.1(b) shows that $|\operatorname{tr} u| = n$ and so $u \in \mathbf{T}$ by the converse of the Cauchy-Schwarz inequality. We can now apply Theorem 4.3. (Alternatively we see that $\mathscr{H} = \bigoplus_{-\infty}^{\infty} \mathscr{H}_m$ where $\mathscr{H}_m = [F(\mathscr{H})S^m\Omega]^-$, $\mathscr{H}_{-m} = [F(\mathscr{H})S^{*m}\,\Omega]^-$ if $m \geqslant 0$ decomposes $\pi|F(\mathscr{H})$ as $\bigoplus_{-\infty}^{\infty} \pi_m$ say. Since

$$\omega_{1/n}[S^m(\cdot)S^{*m}] = \omega_{1/n}(\cdot)n^{-m},$$

it follows that $\{\pi_m : m \leqslant 0\}$ are all equivalent representations. By rotating U if necessary we may assume $U_0 = 1$. Then $U = \bigoplus_{-\infty}^{\infty} t^m$. As $U \in \pi(F(\mathscr{H}))''$, and π_0 and π_{-1} are equivalent, this forces $t = 1$.)

(b) As in (a) suppose $O(u) = \operatorname{Ad} U$ where $U \in O(\mathscr{H})$, $u \in U(\mathscr{H})$. Then $\alpha_t(U) = t^m U$ for some $m \geqslant 0$ say. Letting $A = US^{m*} \in F(\mathscr{H})$ where $S = O(f)$, $\|f\| = 1$, we see that $\omega_0(AA^*) = 1$, $\omega_0(A^*A) = 0$ if $m > 0$. Hence $m = 0$ and $U \in F(\mathscr{H})$. Hence $F(u)$ is inner. Let $\mathscr{A}_m = \left[\bigotimes_1^m \mathscr{K}(\mathscr{H})\right] + \mathbf{C} \bigotimes_1^m 1$. Then $F(\mathscr{H}) = \overline{\bigcup \mathscr{A}_m}$, and $\mathscr{A}_m \subseteq \mathscr{A}_{m+1}$. Then following [11], we may, given $\varepsilon > 0$, find $m > 0$ and a unitary U_m in $\mathscr{A}_m$ such that $\|U - U_m\| < \varepsilon/2$. Hence

$$\|\operatorname{Ad} U - \operatorname{Ad} U_m\| < \varepsilon.$$

Take $y = p \otimes x \in \bigotimes_1^{m+1} \mathscr{K}(\mathscr{H}) \subseteq \mathscr{A}_{m+1}$, for a finite rank projection p in $\bigotimes_1^m \mathscr{K}(\mathscr{H})$, x in $\mathscr{K}(\mathscr{H})$, then we have

$$|\langle[\operatorname{Ad} U(y) - \operatorname{Ad} U_m(y)]\xi,\ \eta\rangle| < \xi\|y\|\cdot\|\xi\|\cdot\|\eta\| \quad \text{for all } \xi,\ \eta \text{ in } \bigotimes_1^{m+1} \mathscr{H}.$$

Letting p increase we see that

$$|\langle[UxU^* - x]\xi_0,\ \eta_0\rangle| < \varepsilon\|x\|\cdot\|\xi_0\|\cdot\|\eta_0\|$$

for all ξ_0, η_0 in $\mathscr{H}$, all x in $\mathscr{K}(\mathscr{H})$, $\varepsilon > 0$. Hence $UxU^* = x$, and so $U \in \mathbf{T}$. Hence $F(u) = \mathrm{id}$, and so $U \in \operatorname{center} F(\mathscr{H}) = \mathbf{C}$. Thus $O(u) = \operatorname{Ad} U = \mathrm{id}$, and so $u = 1$. That $O(u)$ is weakly inner in ω_0, is trivial since ω_0 is pure.

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H. ARAKI
Research Institute for Mathematical Sciences,
Kyoto University,
Kitashirakawa, Sakyo-ku,
Kyoto 606,
Japan.

A. L. CAREY
Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
P.O. Box 4, Canberra, ACT 2600,
Australia.

D. E. EVANS
Mathematics Institute,
University of Warwick,
Coventry, CV4 7AL,
England.

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