A NEW PROOF OF TORELLI'S THEOREM

by

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Let X be a complete, nonsingular curve of genus $g > 1$. Let $\varphi: X \to J(X)$ be a canonical map of X into its Jacobian variety, $J(X)$. Assume φ chosen so that $\varphi(P) = 0$ for some point, $P \varepsilon X$.

If $D = \Sigma d_i Q_i$ is a divisor on X, we define $\varphi(D) = \Sigma d_i \varphi(Q_i)$. The image under φ of the positive divisors of degree $\leq r$ on X will be denoted by W^r, and we extend this definition by setting $W^0 = \{0\}$.

It is known[2] that W^1 is birationally equivalent to X, and that W^{g-1} determines the canonical polarization of $J(X) = W^g$. The object of this paper is to prove that W^1 is determined up to a translation and reflection by $J(X)$ and W^{g-1}, (i.e., X is determined up to a birational equivalence by the same data).

A classical version of this theorem was proved by Torelli.[4] Weil[5] gave a modern proof, valid in the abstract case, based on an idea of Andreotti. Other abstract proofs were later given by Matsusaka[3] and Andreotti.[1]

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The proof to be given here is based on a modification of two of Weil's lemmas which enables us to recover Torelli's theorem as a combinatorial consequence of the Riemann-Roch theorem and Abel's theorem.

We begin by proving four preliminary lemmas of which the second and fourth may be characterized as modifications of Weil's Hilfssätze 3 and 1, respectively. Lemmas 2, 3, and 4 admit generalizations which, however, are not needed for our purposes.

We denote, as usual, by W_a^r the translate of W^r by an element $a \varepsilon J(X)$. Following Weil,[5] we denote by $\left(W_a^r\right)^*$ the image of W_a^r under the map $u \rightarrow - u + \varphi(Z)$ where Z is a canonical divisor on X. We recall[2] that the sets W_a^r and $\left(W_a^r\right)^*$ are subvarieties of J(X).

Our first lemma is a known result which we prove for convenience:

<u>Lemma 1</u>

$$\left(W_a^{g-1}\right)^* = W_{-a}^{g-1}$$

<u>Proof</u>: Given a positive divisor, D, of degree (g-1), there exists a positive divisor D′, of degree (g-1) such that $D+D' \sim Z$, where $\sim$ denotes linear equivalence. By Abel's theorem

$$\varphi(D) - a = - \varphi(D') + \varphi(Z) - a$$

As the left-hand side traverses W_{-a}^{g-1} the right-hand side traverses $\left(W_a^{g-1}\right)^*$, and conversely.

<u>Lemma 2</u>

<u>Let</u>

$$0 \leq r \leq g-1 .$$

<u>Then</u>

$$W_a^r \subset W_b^{g-1} \longleftrightarrow a \in W_b^{g-1-r} .$$

<u>Proof</u>: The implication from right to left is trivial. Assume now that $W_a^r \subset W_b^{g-1}$. This means that for every positive divisor, D, of degree r, there is a positive divisor, $\hat{D}$, of degree g-1, such that $\varphi(D) + a = \varphi(\hat{D}) + b$. In particular, there is a positive divisor, A, of degree g-1, such that $a = \varphi(A) + b$. Hence, $\varphi(D) + \varphi(A) = \varphi(\hat{D})$, and, by Abel's theorem

$$D+A \sim \hat{D}+rP .$$

Let A′ and $\hat{D}'$ be positive divisors of degree g-1 such that A+A′ and $\hat{D}+\hat{D}'$ are canonical divisors. Then

$$D+\hat{D}' \sim A'+rP .$$

Since an equivalence of this form must hold for all positive divisors, D, of degree r, it follows* that $\ell(A'+rP) \geq r+1$.

*By $\ell(D)$ we denote the dimension of the (linear) space of functions whose divisors are $\geq -D$.

By the Riemann-Roch theorem it follows that $\ell(Z-A'-rP) \geq 1$. Hence, there is a positive divisor, $\hat{A}$, of degree g-l-r such that $A'+rP+\hat{A} \sim Z$, whence $\varphi(A) = \varphi(\hat{A})$. But then

$$a = \varphi(\hat{A}) + b \in W_b^{g-1-r} .$$

Lemma 3

Let

$$0 \leq r \leq g-1 .$$

Then

$$W^{g-1-r} = \cap \left\{W_{-u}^{g-1} : u \in W^r\right\}$$

and

$$(W^{g-1-r})^* = \cap \left\{W_{+u}^{g-1} : u \in W^r\right\} .$$

Proof: By Lemma 2,

$$W^{g-1-r} \subset W_{-u}^{g-1} \longleftrightarrow W_u^{g-1-r} \subset W^{g-1} \longleftrightarrow u \in W^r .$$

Hence

$$W^{g-1-r} \subset \cap \left\{W_{-u}^{g-1} : u \in W^r\right\} .$$

On the other hand, if $v \in W_{-u}^{g-1}$ for all $u \in W^r$, then $u \in W_{-v}^{g-1}$ for all $u \in W^r$, whence $W^r \subset W_{-v}^{g-1}$ and $v \in W^{g-1-r}$, by Lemma 2. This proves the first formula, and the second formula follows from the equation

$$\cap \left\{W_{+u}^{g-1} : u \varepsilon W^r\right\} = \cap \left\{\left(W_{-u}^{g-1}\right)^* : u \varepsilon W^r\right\}$$

$$= \left(\cap \left\{W_{-u}^{g-1} : u \varepsilon W^r\right\}\right)^* .$$

Lemma 4

Let

$$0 \leq r \leq g-1 .$$

Let a and b be related by an equation, $b = a+x-y$, where $x \varepsilon W^1$ and $y \varepsilon W^{g-1-r}$. Then either $W_a^{r+1} \subset W_b^{g-1}$, or else

$$W_a^{r+1} \cap W_b^{g-1} = W_{a+x}^r \cup S$$

where

$$S = W_a^{r+1} \cap \left(W_{y-a}^{g-2}\right)^* .$$

Proof: By assumption, $x = \varphi(R)$, $y = \varphi(\hat{R})$ and $\varphi(R) + a = \varphi(\hat{R}) + b$, where R and $\hat{R}$ are positive divisors of degrees 1 and g-1-r, respectively. If R is a point of $\hat{R}$, we get an equation $a = \varphi(R') + b$, where $\deg(R') = g-2-r$. But then $a \varepsilon W_b^{g-2-r}$ and $W_a^{r+1} \subset W_b^{g-1}$. Hence we assume that R is not a point of $\hat{R}$.

Let $u \varepsilon W_a^{r+1} \cap W_b^{g-1}$. Then there are positive divisors, D and $\hat{D}$, of degrees r+1 and g-1, respectively, such that $u = \varphi(D) + a = \varphi(\hat{D}) + b$. Hence

$$D+\hat{R} \sim \hat{D}+R .$$

If $D+\hat{R} = \hat{D}+R$, R must be a point of D and $u = \varphi(D) + a = \varphi(D') + \varphi(R) + a$, where $\deg(D') = r$. Then $u \varepsilon W^r_{a+x}$.

If $D+\hat{R} \neq \hat{D}+R$, then $\ell(D+\hat{R}) \geq 2$, and, given any point, $Q \varepsilon X$, there is a positive divisor, $\hat{Q}$, of degree g-1, such that $D+\hat{R} \sim Q+\hat{Q}$. Then

$$u = \varphi(D) + a = \varphi(\hat{Q}) + \varphi(Q) - \varphi(\hat{R}) + a ,$$

whence

$$u \ \varepsilon \cap \left\{W^{g-1}_{a-y+v} : v \varepsilon W^1\right\} = \left(W^{g-2}_{y-a}\right)^* .$$

Since

$$\left(W^{g-2}_{y-a}\right)^* \subset \left(W^{g-1}_{y-a-x}\right)^* = W^{g-1}_b ,$$

the proof is completed.

Theorem

Let $\varphi : X \to J(X)$ be a canonical map of a complete, nonsingular curve, X, of genus $g > 1$, into its Jacobian variety J(X). Then $W^1 = \varphi(X)$ is determined up to a translation and reflection by the canonical polarization of J(X).

Proof: By a translation, if necessary, we may normalize φ such that $\varphi(P) = 0$ for some point, $P \varepsilon X$. Let Y be a second curve with the same Jacobian variety, J(X), and denote by V^r the image of the set of positive divisors of degree $\leq$ r on Y under the (normalized) canonical map $\psi: Y \to J(X)$. The theorem will be proved by showing that if V^{g-1} is a translate of W^{g-1} (i.e., if the canonical polarizations are the same) then V^1 is a translate of W^1 or of $(W^1)^*$.

Let r be the smallest integer such that $V^1 \subset W_a^{r+1}$ or $V^1 \subset \left(W_a^{r+1}\right)^*$ for some a. The theorem will be proved if we can show that r = 0. Assume to the contrary that $r \geq 1$. (Clearly, $r < g-1$.) Assume, changing notation if necessary, that $V^1 \subset W_a^{r+1}$. Choose $x \varepsilon W^1$, $y \varepsilon W^{g-1-r}$, and set $b = a+x-y$. Then, unless $W_a^{r+1} \subset W_b^{g-1}$, we have

$$V^1 \cap W_b^{g-1} = V^1 \cap W_b^{g-1} \cap W_a^{r+1} = \left(V^1 \cap W_{a+x}^r\right) \cup (V^1 \cap S)$$

in the notation of Lemma 4. Note that, a being given, W_{a+x}^r depends only on the choice of x, and S depends only on the choice of y.

We shall first show that for a fixed x, $V^1 \not\subset W_b^{g-1}$ for almost all choices of y, and hence $W_a^{r+1} \not\subset W_b^{g-1}$ for the same y.

As y varies over W^{g-1-r}, -b varies over $W_{-(a+x)}^{g-1-r}$. By assumption, there is a constant k, such that $V_k^{g-1} = W^{g-1}$. Hence, $V^1 \subset W_b^{g-1} \longleftrightarrow V^1 \subset V_{b+k}^{g-1} \longleftrightarrow -b \varepsilon V_k^{g-2}$. Thus the set of b for which $V^1 \subset W_b^{g-1}$ is given by $-b \varepsilon V^{g-2} \cap W_{-(a+x)}^{g-1-r}$.

Now, if $V^1 \subset W_b^{g-1}$ for all $-b \varepsilon W_{-(a+x)}^{g-1-r}$, then $V^1 \subset W_{a+x}^r$ by Lemma 3. This contradicts the assumption on r. Hence $W_{-(a+x)}^{g-1-r} \not\subset V_k^{g-2}$, and the intersection of these sets is a lower dimensional subset of $W_{-(a+x)}^{g-1-r}$.

We now return to consider the intersection

$$V^1 \cap W_b^{g-1} = \left(V^1 \cap W_{a+x}^r\right) \cup (V^1 \cap S) \ .$$

It is well known[2] that if $V^1 \not\subset W_b^{g-1}$, then there is a unique positive divisor, D(b), of degree g on Y, such that

$$\psi(D(b)) = b+c \qquad (1)$$

where c is a constant, independent of b, and the points of D(b) are the preimages of the points of the intersection $V^1 \cap W_b^{g-1}$ under ψ.

We show first that $V^1 \cap W_{(a+x)}^r$ contains at most one point. If not, then as -b varies over almost all points of $W_{-(a+x)}^{g-1-r}$ (for fixed x), D(b) will contain at least two fixed points, and hence $\psi(D(b))$ varies over a translate of V^{g-2}. By Eq. (1) we should then have an inclusion of $(W^{g-1-r})^*$ in a translate of V^{g-2}, say $(W^{g-1-r})^* \subset V_d^{g-2}$. But then

$$\cap \left\{V_{k-u}^{g-1} : u \varepsilon V_d^{g-2}\right\} \subset \cap \left\{W_{-u}^{g-1} : u \varepsilon (W^{g-1-r})^*\right\}$$

and, using Lemma 3, we get an inclusion of V^1 in a translate of $(W^r)^*$, contradicting the assumption on r.

Keeping y fixed and varying x, we see by Eq. (1) that $V^1 \cap W^r_{a+x}$ must contain at least one point, and hence it contains exactly one point.*

It is now easily seen that we can find $x, x' \varepsilon W^1$ and $y \varepsilon W^{g-1-r}$ such that $D(a+x-y) = Q+\hat{D}$ and $D(a+x'-y) = Q'+\hat{D}$ where $Q, Q' \varepsilon Y$ and $\hat{D}$ is a positive divisor of degree g-1 on Y not containing Q or Q'. By Eq. (1), $\varphi(Q) - \varphi(Q') = x-x'$, and hence W^1 has two distinct points in common with some translate of V^1. Now, if $x, x' \varepsilon W^1$, then $W_{-x}^{g-1} \cap W_{-x'}^{g-1} = W^{g-2} \cup \left(W^{g-2}_{x+x'}\right)^*$ by Lemma 4. By Lemma 3 we now get an inclusion of some translate of V^{g-2} in W^{g-2} or in $(W^{g-2})^*$, whence, again by Lemma 3, we get an inclusion of some translate of V^1 in W^1 or $(W^1)^*$. This completes the proof.

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* Which, by the preceding argument, occurs in D(b) with multiplicity 1, for almost all choices of y.

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