Talk 1: An Introduction to Graph C^*-algebras

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Some Terminology

A graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge.

A *path* $e_1 \dots e_n$ is a sequence of edges with $r(e_i) = s(e_{i+1})$. A *cycle* is a path with $r(e_n) = s(e_1)$, and we call $s(e_1)$ the *base point* of this cycle.

A *sink* is a vertex that emits no edges; i.e., $s^{-1}(v) = \emptyset$. We write E^0_{sinks} for the set of sinks.

An *infinite emitter* is a vertex that emits an infinite number of edges; i.e., $s^{-1}(v)$ is infinite. We write E^0_{inf} for the set of infinite emitters.

A *regular vertex* is a vertex that emits a finite and nonzero number of edges; i.e., $0 < |s^{-1}(v)| < \infty$. We write E^0_{reg} for the set of regular vertices.

We say a graph is *row-finite* if the graph has no infinite emitters.

Definition

If $E = (E^0, E^1, r, s)$ is a directed graph consisting of a countable set of vertices E^0, a countable set of edges E^1, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge, then $C^*(E)$ is defined to be the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges that satisfy

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
2. $p_v = \sum_{s(e)=v} s_e s_e^*$ when $0 < |s^{-1}(v)| < \infty$
3. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$.

NOTE: At the beginning we'll restrict to the row-finite case.

NOTE: For row-finite graphs, (2) $\implies$ (3).

(1) Not only does the graph summarize the relations that the generators satisfy, but also the C^*-algebraic properties of $C^*(E)$ are encoded in the graph E.

(2) Also, graph C^*-algebras are fairly tractable. Their structure can be deduced and their invariants can be computed.

(3) Graph C^*-algebras include many C^*-algebras.

Up to isomorphism, graph C^*-algebras include:

- All Cuntz algebras and all Cuntz-Krieger algebras
- All finite-dimensional C^*-algebras
- $C(\mathbb{T})$, $\mathcal{K}(H)$, $M_n(C(\mathbb{T}))$, $\mathcal{T}$, and certain quantum algebras

Up to Morita Equivalence, graph C^*-algebras include:

- All AF-algebras
- All Kirchberg algebras with free K_1-group

THE STANDARD GAUGE ACTION

By the universal property of $C^*(E)$, there exists an action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ with

$$\gamma_z(s_e) = zs_e \qquad \text{and} \qquad \gamma_z(p_v) = p_v$$

for all $e \in E^1$ and $v \in E^0$.

We say an ideal $I \triangleleft C^*(E)$ is *gauge invariant* if $\gamma_z(I) \subseteq I$ for all $z \in \mathbb{T}$.

Two technical theorems:

Theorem (Gauge-Invariant Uniqueness)

Let E be a directed graph and let $\rho : C^*(E) \to B$ be a $*$-homomorphism between C^*-algebras. Also let γ denote the standard gauge action on $C^*(E)$. If there exists an action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for each $z \in \mathbb{T}$, and if $\rho(p_v) \neq 0$ for all $v \in E^0$, then ρ is injective.

Definition: An *exit* for a cycle $e_1 \dots e_n$ is an edge f with $s(f) = s(e_i)$ but $f \neq e_i$ for some i.

Condition (L): Every cycle has an exit.

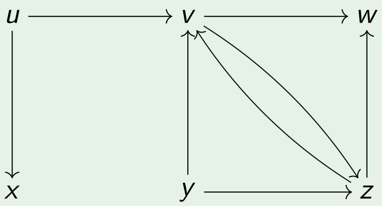
Theorem (Cuntz-Krieger Uniqueness)

Let E be a directed graph satisfying Condition (L) and let $\rho : C^*(E) \to B$ be a $*$-homomorphism between C^*-algebras. If $\rho(p_v) \neq 0$ for all $v \in E^0$, then ρ is injective.

Let $E = (E^0, E^1, r, s)$ be a graph. A subset $H \subseteq E^0$ is *hereditary* if for any $e \in E^1$ we have $s(e) \in H$ implies $r(e) \in H$. A hereditary subset $H \subseteq E^0$ is said to be *saturated* if whenever $v \in E^0$ is a regular vertex with $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$, then $v \in H$.

If $H \subseteq E^0$ is a hereditary set, the *saturation of H* is the smallest saturated subset $\overline{H}$ of E^0 containing H.

Example



The set $X = \{v, w, z\}$ is hereditary but not saturated. The set $H = \{v, w, y, z\}$ is both saturated and hereditary. We see that $\overline{X} = H$.

Theorem

Let $E = (E^0, E^1, r, s)$ *be row-finite.*

$$I_H := \text{ideal in } C^*(E) \text{ generated by } \{p_v : v \in H\}$$

(a) $H \mapsto I_H$ *is an isomorphism from the lattice of saturated hereditary subsets of* E *onto the lattice of gauge-invariant ideals of* $C^*(E)$.)

(b) *If* H *is saturated hereditary, and we let* $E \setminus H$ *be the subgraph of* E *whose vertices are* $E^0 \setminus H$ *and whose edges are* $E^1 \setminus r^{-1}(H)$, *then* $C^*(E)/I_H$ *is isomorphic to* $C^*(E \setminus H)$.

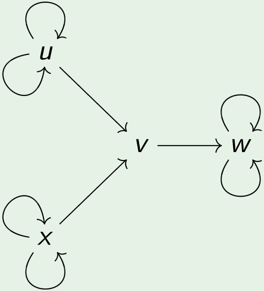
(c) *If* X *is any hereditary subset of* E^0, *then* $I_X = I_{\overline{X}}$. *If we let* E_X *denote the subgraph of* E *with vertices* X *and edges* $s^{-1}(X)$, *then* $C^*(E_X)$ *is isomorphic to the subalgebra*

$$C^*(\{s_e, p_v : e \in s^{-1}(X) \text{ and } v \in X\}),$$

and this subalgebra is a full corner of the ideal I_X.

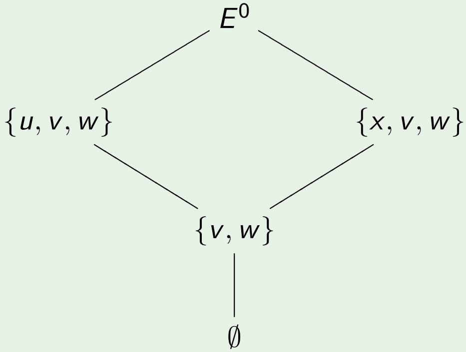
Example

Let E be the graph



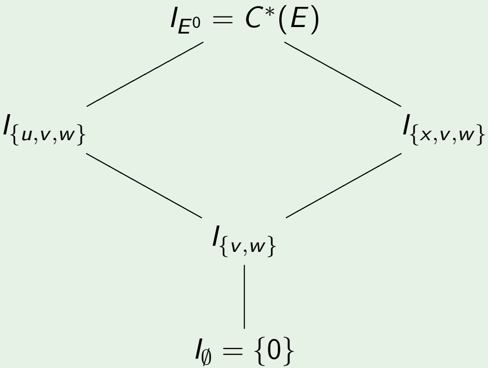
Example

Then the saturated hereditary subsets of E are



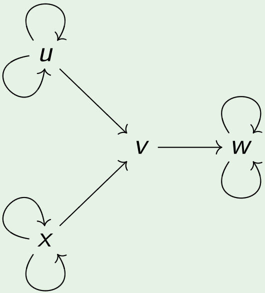
Example

and gauge-invariant ideals of $C^*(E)$ are



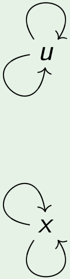
Example

Let E be the graph

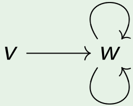


Let $H = \{v, w\}$. Then

$E \setminus H$



E_H



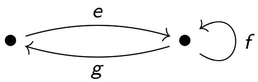
$C^*(E)/I_H \cong C^*(E \setminus H)$ $\qquad$ I_H is Morita equivalent to $C^*(E_H)$.

Note: For surjectivity of $H \mapsto I_H$, we need to apply the GIUT to $E \setminus H$. If $E \setminus H$ satisfies Condition (L) for all H, then we could instead use the CKUT and show all ideals are gauge-invariant.

Definition

A *simple cycle* in a graph E is a cycle $\alpha = \alpha_1 \dots \alpha_n$ with the property that $s(\alpha_i) \neq s(\alpha_1)$ for $i \in \{2, 3, \dots, n\}$.

Condition (K): No vertex in E is the base point of exactly one simple cycle; that is, every vertex in E is either the base point of no cycles or of more than one simple cycle.



The above graph satisfies Condition (K).
Note: Condition (K) implies Condition (L).

Theorem

If E is a graph, then E satisfies Condition (K) if and only if for every saturated hereditary subset H of E^0 the subgraph $E \setminus H$ satisfies Condition (L).

Theorem

A graph E satisfies Condition (K) if and only if all ideals of $C^*(E)$ are gauge invariant.

(Note: In the earlier example we considered, the lattice of gauge-invariant ideals that we described consists of *all* the ideals.)

Definition

For $v, w \in E^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. In this case we say that v *can reach* w.

Definition

We say that a graph E is *cofinal* if for every $v \in E^0$ and every infinite path $\alpha \in E^\infty$, there exists $i \in \mathbb{N}$ for which $v \geq s(\alpha_i)$.

Theorem

Let E be a row-finite graph with no sinks. Then $C^*(E)$ is simple if and only if E satisfies Condition (L) and E is cofinal.

A C^*-algebra is an *AF-algebra* (AF stands for *approximately finite-dimensional*) if it can be written as the closure of the increasing union of finite-dimensional C^*-algebras; or, equivalently, if it is the direct limit of a sequence of finite-dimensional C^*-algebras.

Theorem

(Kumjian, Pask, Raeburn) If E is a row-finite graph, then $C^*(E)$ is AF if and only if E has no cycles.

A simple C^*-algebra A is *purely infinite* if every nonzero hereditary subalgebra of A contains an infinite projection. (The definition of purely infinite for non-simple C^*-algebra is more complicated.)

Theorem

(Kumjian, Pask, and Raeburn) If E is a row-finite graph, then every nonzero hereditary subalgebra of $C^*(E)$ contains an infinite projection if and only if E satisfies Condition (L) and every vertex in E connects to a cycle.

THE DICHOTOMY

Theorem (The Dichotomy for Simple Graph Algebras)

Let E be a row-finite graph. If $C^*(E)$ is simple, then either

1. *$C^*(E)$ is an AF-algebra if E contains no cycles; or*
2. *$C^*(E)$ is purely infinite if E contains a cycle.*

NON-ROW-FINITE GRAPHS

Up until now all of our graphs have been row-finite. How do we deal with arbitrary graphs?

We will use the notation

$$v \xrightarrow{(\infty)} w$$

to indicate that there are a countably infinite number of edges from v to w.

In order to *desingularize* graphs, we will need to remove sinks and infinite emitters.

Definition

If E is a graph and v_0 is a sink in E, then by *adding a tail at* v_0 we mean attaching a graph of the form

$$v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots$$

to E at v_0.

Definition

If E is a graph and v_0 is an infinite emitter in E, then by *adding a tail at* v_0 we mean performing the following process: We first list the edges $g_1, g_2, g_3, \ldots$ of $s^{-1}(v_0)$. Then we add a graph of the form

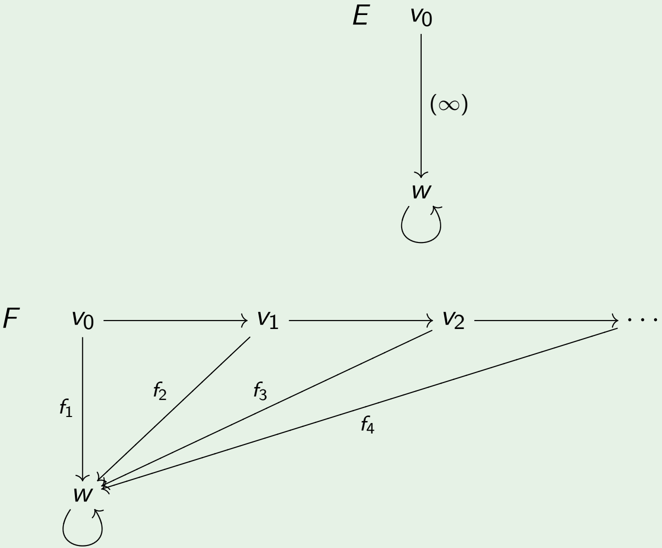
$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

to E at v_0, remove the edges in $s^{-1}(v_0)$, and for every $g_j \in s^{-1}(v_0)$ we draw an edge f_j from v_{j-1} to $r(g_j)$.

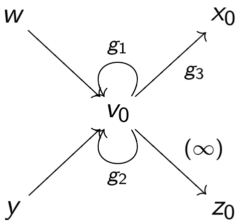
Note: Desingularization is not unique.

Example

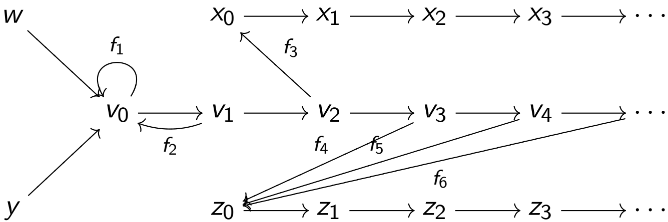
Here is an example of a graph E and a desingularization F of E.



Suppose E is the following graph:

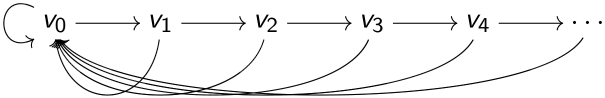


Label the edges from v_0 to z_0 as $\{g_4, g_5, g_6, \ldots\}$. Then a desingularization of E is given by the following graph F.



If E is the $\mathcal{O}_\infty$ graph shown here

then a desingularization is given by:



Theorem

Let E be a graph. If F is a desingularization of E and p_{E^0} is the projection in $M(C^*(F))$ defined by $p_{E^0} := \sum_{v \in E^0} p_v$, then $C^*(E)$ is isomorphic to the corner $p_{E^0} C^*(F) p_{E^0}$, and this corner is full.

The advantage of the process of desingularization is that it is very concrete, and it allows us to use the row-finite graph F to see how the properties of $C^*(E)$ are reflected in the graph E. We will see examples of this in the following, as we show how to extend results for C^*-algebras of row-finite graphs to general graph algebras.

Theorem

Let E be a graph. The graph algebra $C^*(E)$ is an AF-algebra if and only if E has no cycles.

Proof.

Let F be a desingularization of E. Then

$$\begin{aligned} C^*(E) \text{ is AF} &\iff C^*(F) \text{ is AF} \\ &\iff F \text{ has no cycles} \\ &\iff E \text{ has no cycles.} \end{aligned}$$

□

Theorem

Let E be a graph. If E satisfies Condition (L) and every vertex in E connects to a cycle in E, then there exists an infinite projection in every nonzero hereditary subalgebra of $C^*(E)$.

Proof.

Let F be a desingularization of E. Then

E satisfies Condition (L) and every vertex in E connects to a cycle

$\Longrightarrow$ F satisfies Condition (L) and every vertex in F connects to a cycle

$\Longrightarrow$ there is an infinite projection in every nonzero hereditary subalgebra of $C^*(F)$

$\Longrightarrow$ there is an infinite projection in every nonzero hereditary subalgebra of $C^*(E)$.

□

Theorem

If E is a graph, then $C^*(E)$ is simple if and only if E has the following four properties:

1. *E satisfies Condition (L),*
2. *E is cofinal,*
3. *if $v, w \in E^0$ with v a sink, then $w \geq v$, and*
4. *if $v, w \in E^0$ with v an infinite emitter, then $w \geq v$.*

Proof.

Let F be a desingularization of E. Then

$$\begin{aligned} & C^*(E) \text{ is simple} \\ \iff & C^*(F) \text{ is simple} \\ \iff & F \text{ satisfies Condition (L) and is cofinal} \\ \iff & E \text{ satisfies Condition (L), is cofinal, and each vertex} \\ & \quad \text{can reach every sink and every infinite emitter.} \end{aligned}$$

Theorem (The Dichotomy for Simple Graph Algebras)

Let E be a graph. If $C^*(E)$ is simple, then either

1. *$C^*(E)$ is an AF-algebra if E contains no cycles; or*
2. *$C^*(E)$ is purely infinite if E contains a cycle.*

What about ideals when the graph is not row-finite?
Let E be a graph that satisfies Condition (K). Then

$$H \mapsto I_H := \text{the ideal generated by } \{p_v : v \in H\}$$

is still injective, using the same proof as before.
However, it is no longer true that this map is surjective. The reason the proof for row-finite graphs no longer works is that if I is an ideal, then $\{s_e + I, p_v + I\}$ will not necessarily be a Cuntz-Krieger $E \setminus H$-family for the graph $E \setminus H$. (And, consequently, it is sometimes not true that $C^*(E)/I_H \cong C^*(E \setminus H)$.)
To describe an ideal in $C^*(E)$ we will need a saturated hereditary subset and one other piece of information. Loosely speaking, this additional piece of information tells us how close $\{s_e + I, p_v + I\}$ is to being a Cuntz-Krieger $E \setminus H$-family.

Given a saturated hereditary subset $H \subseteq E^0$, we define the *breaking vertices* of H to be the set

$$B_H := \{v \in E^0 : v \text{ is an infinite-emitter and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

We see that B_H is the set of infinite-emitters that point to a finite number of vertices not in H. Also, since H is hereditary, B_H is disjoint from H. Fix a saturated hereditary subset H of E, and let $S \subseteq B_H$. Define

$$I_{(H,S)} := \text{the ideal in } C^*(E) \text{ generated by } \{p_v : v \in H\} \cup \{p_{v_0}^H : v_0 \in S\},$$

where

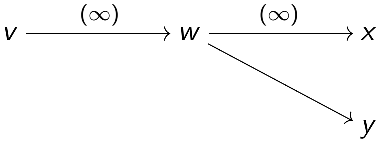
$$p_{v_0}^H := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e)\notin H}} s_e s_e^*.$$

Note that the definition of B_H ensures that the sum on the right is finite.

Definition

We say that (H, S) is an *admissible pair* for E if H is a saturated hereditary subset of vertices of E and $S \subseteq B_H$. We order admissible pairs by defining $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$.

Let E be the graph



Then the saturated hereditary subsets of E are

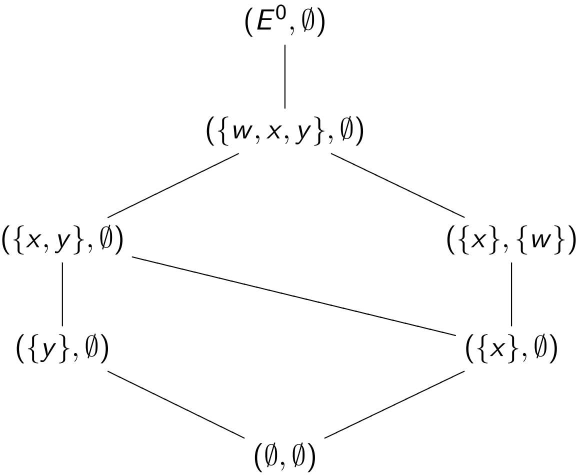
$$E^0, \{w, x, y\}, \{x, y\}, \{y\}, \{x\}, \text{and } \emptyset.$$

Also $B_{\{x\}} = \{w\}$, and $B_H = \emptyset$ for all other H. The admissible pairs of E are:

$$(E^0, \emptyset), (\{w, x, y\}, \emptyset), (\{x, y\}, \emptyset), (\{y\}, \emptyset),$$

$$(\{x\}, \{w\}), (\{x\}, \emptyset), (\emptyset, \emptyset)$$

These admissible pairs are ordered in the following way.



Theorem

Let E be a graph. The map $(H, S) \mapsto I_{(H,S)}$ is a lattice isomorphism from admissible pairs for E onto the gauge-invariant ideals of $C^*(E)$. (When E satisfies Condition (K) all ideals are gauge invariant, and this map is onto the lattice of ideals of $C^*(E)$.

We'll sketch a proof of this using desingularization.

Lemma

Suppose A is a C^*-algebra, p is a projection in the multiplier algebra $M(A)$, and pAp is a full corner of A. Then the map $I \mapsto pIp$ is an order-preserving bijection from the ideals of A to the ideals of pAp. Moreover, this map restricts to a bijection from gauge-invariant ideals of A onto the gauge-invariant ideals of pAp.

Let E be a graph and F a desingularization. Also let (H, S) be an admissible pair for E.
We define

$$\tilde{H} := H \cup \{v_n \in F^0 : v_n \text{ is on a tail added} \tag{1}$$
$$\text{to a vertex in } H\} \tag{2}$$

Now for each $v_0 \in S$ let N_{v_0} be the smallest nonnegative integer such that $r(f_j) \in H$ for all $j > N_{v_0}$.
Define

$$T_{v_0} := \{v_n : v_n \text{ is on the tail added} \tag{3}$$
$$\text{to } v_0 \text{ and } n \geq N_{v_0}\} \tag{4}$$

and define

$$H_S := \tilde{H} \cup \bigcup_{v_0 \in S} T_{v_0}.$$

Lemma

The map $(H, S) \mapsto H_S$ is an order-preserving bijection from the lattice of admissible pairs of E onto the lattice of saturated hereditary subsets of F.

Lemma

Let E be a graph and let F be a desingularization of E. Let p_{E^0} be the projection in $M(C^*(F))$ defined by $p_{E^0} = \sum_{v \in E^0} p_v$, and identify $C^*(E)$ with $p_{E^0} C^*(F) p_{E^0}$. If H is a saturated hereditary subset of E^0 and $S \subseteq B_H$, then then

$$p_{E^0} I_{H_S} p_{E^0} = I_{(H,S)}.$$

This shows that the following diagram commutes

$$\begin{array}{ccc} \text{admissible pairs in } E & \xrightarrow{(H,S)\mapsto I_{(H,S)}} & \text{ideals in } C^*(E) \\ {\scriptstyle (H,S) \mapsto H_S} \Big\downarrow & & \Big\uparrow {\scriptstyle I \mapsto pIp} \\ \text{sat. her. subsets of } F & \xrightarrow[H\mapsto I_H]{} & \text{ideals in } C^*(F). \end{array}$$

and we have our result.

The ideals $I_{(H,S)}$ are precisely the gauge-invariant ideals in $C^*(E)$.

However, the quotient $C^*(E)/I_{(H,S)}$ is not necessarily isomorphic to $C^*(E \setminus H)$ because the collection $\{s_e + I_{(H,S)}, p_v + I_{(H,S)}\}$ may fail to satisfy the third Cuntz-Krieger relation at breaking vertices for H.

Nonetheless, $C^*(E)/I_{(H,S)}$ is isomorphic to $C^*(F_{H,S})$, where $F_{H,S}$ is the graph defined by

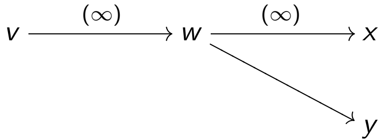
$$F^0_{H,S} := (E^0 \backslash H) \cup \{v' : v \in B_H \backslash S\}$$

$$F^1_{H,S} := \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \backslash S\}$$

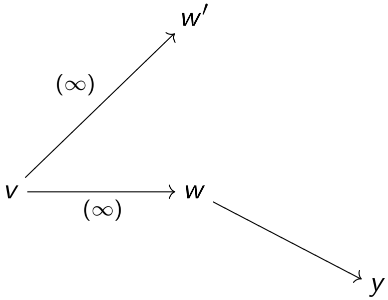
and r and s are extended by $s(e') = s(e)$ and $r(e') = r(e)'$.

(Note: $F_{(H,B_H)} = E \setminus H$.)

Let E be the graph



Let $(H, S) = (\{x\}, \emptyset)$. (Note: $B_{\{x\}} = \{w\}$.)
Then $F_{(H,S)}$ is the graph



and $C^*(E)/I_{(H,S)} \cong C^*(F_{(H,S)})$.