Talk 2: More on Graph C^*-algebras

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K-theory is an important invariant for C^*-algebras. Moreover, in certain situations K-theory classifies C^*-algebras up to Morita equivalence and up to isomorphism.

One remarkable, and very useful, aspect of graph C^*-algebras is that we can compute the K-theory in a concrete manner. Also, in many situations we can determine the range of this invariant.

Let A be a unital C^*-algebra.

Definition

Let $\operatorname{Proj} M_n(A)$ be the set of projections in $M_n(A)$. Identifying $p \in \operatorname{Proj} M_n(A)$ with the projection $p \oplus 0$ in $\operatorname{Proj} M_{n+1}(A)$ we may view $\operatorname{Proj} M_n(A)$ as a subset of $\operatorname{Proj} M_{n+1}(A)$. We let

$$\operatorname{Proj}_\infty(A) = \bigcup_{n=1}^{\infty} \operatorname{Proj} M_n(A).$$

For $p, q \in \operatorname{Proj}_\infty(A)$ we write $p \sim q$ if there exists $u \in \operatorname{Proj}_\infty(A)$ with $p = uu^*$ and $q = u^*u$.

We let $[p]_0$ denote the equivalence class of $p \in \operatorname{Proj}_\infty(A)$. We define an addition on these equivalence classes by setting $[p]_0 + [q]_0$ equal to $\left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]_0$. Then $\operatorname{Proj}_\infty(A)/\sim$ is an abelian semigroup. We define $K_0(A)$ as its *Grothendieck group*; that is $K_0(A)$ is the abelian group of formal differences

$$K_0(A) := \{[p]_0 - [q]_0 : p, q \in \operatorname{Proj}_\infty(A)\}.$$

Definition

The group $K_1(A)$ is defined using the groups $U(M_n(A))$ of unitary elements in $M_n(A)$. We embed $U(M_n(A))$ into $U(M_{n+1}(A))$ by $u \mapsto u \oplus 1$. We then let

$$U_\infty(A) := \bigcup_{n=1}^{\infty} U(M_n(A)).$$

We define an equivalence relation on $U_\infty(A)$ as follows: If $u \in U_m(A)$ and $v \in U_n(A)$, we write $u \sim v$ if there is a natural number $k \geq \max\{m, n\}$ such that $\left(\begin{smallmatrix} u & 0 \\ 0 & 1_{k-n} \end{smallmatrix}\right)$ is homotopic to $\left(\begin{smallmatrix} v & 0 \\ 0 & 1_{k-m} \end{smallmatrix}\right)$ in $U_k(A)$ (i.e., there exists a continuous map $h : [0,1] \to U_k(A)$ such that $h(0) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1_{k-n} \end{smallmatrix}\right)$ and $h(1) = \left(\begin{smallmatrix} v & 0 \\ 0 & 1_{k-m} \end{smallmatrix}\right)$. We denote the equivalence class of $u \in U_\infty(A)$ by $[u]_1$. We define $K_1(A)$ to be

$$K_1(A) := \{[u]_1 : u \in U_\infty(A)\}$$

with addition given by $[u]_1 + [v]_1 := [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]_1$. It is true (but not obvious) that $K_1(A)$ is an abelian group.

The K-groups $K_0(A)$ and $K_1(A)$ can also be defined when A is nonunital. If $\phi : A \to B$ is a homomorphism between C^*-algebras, then ϕ induces homomorphisms $\phi_n : M_n(A) \to M_n(B)$ by $\phi((a_{ij})) = (\phi(a_{ij}))$. Since the ϕ_n's map projections to projections and unitaries to unitaries, they induce

$$K_0(\phi) : K_0(A) \to K_0(B)$$

and

$$K_1(\phi) : K_1(A) \to K_1(B).$$

This process is *functorial*: the identity homomorphism induces the identity map on K-groups, and $K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi)$ for $i = 0, 1$.

An *ordered abelian group* (G, G^+) is an abelian group G together with a distinguished subset $G^+ \subseteq G$ satisfying

(i) $G^+ + G^+ \subseteq G^+$,

(ii) $G^+ \cap (-G^+) = \{0\}$,

(iii) $G^+ - G^+ = G$.

We call G^+ the *positive cone* of G, and it allows us to define an ordering on G by setting $g_1 \leq g_2$ if and only if $g_2 - g_1 \in G^+$.
We set

$$K_0(A)^+ := \{[p]_0 : p \in \mathrm{Proj}_\infty(A)\}.$$

If A is an AF C^*-algebra, then $(K_0(A), K_0(A)^+)$ is an ordered abelian group.

Remark

If E is a graph and $v \in E^0$ is a vertex that is neither a sink nor an infinite emitter, then $p_v = \sum_{s(e)=v} s_e s_e^*$, and in $K_0(C^*(E))$ we have

$$\begin{aligned}
[p_v]_0 &= \left[\sum_{s(e)=v} s_e s_e^* \right]_0 \\
&= \sum_{s(e)=v} [s_e s_e^*]_0 \\
&= \sum_{s(e)=v} [s_e^* s_e]_0 \\
&= \sum_{s(e)=v} [p_{r(e)}]_0.
\end{aligned}$$

It turns out that $K_0(C^*(E))$ is generated by the collection $\{[p_v]_0 : v \in E^0\}$ and this collection is subject only to the above relations.

Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sinks. The *vertex matrix* of E is the (possibly infinite) $E^0 \times E^0$ matrix A_E whose entries are the non-zero integers

$$A_E(v, w) := \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}.$$

Let E be a row-finite graph. Then each row of the matrix A_E contains a finite number of nonzero entries, and each column of the transpose A_E^t contains a finite number of nonzero entries.
Therefore, we have a map

$$A_E^t : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$$

defined by left multiplication.

Theorem

Let $E = (E^0, E^1, r, s)$ be a row-finite graph with no sinks. If A_E is the vertex matrix of E, and $A_E^t - I : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$ by left multiplication, then

$$K_0(C^*(E)) \cong \operatorname{coker}(A_E^t - I)$$

via an isomorphism taking $[p_v]_0$ to $[\delta_v]$ for each $v \in E^0$, and

$$K_1(C^*(E)) \cong \ker(A_E^t - I).$$

Moreover, $K_0(C^*(E))^+$ is identified with $\left\{ \sum_{k=1}^N n_k[\delta_{v_k}] : n_k \in \mathbb{N} \right\}$ in $\operatorname{coker}(A_E^t - I)$.

Note: For any graph E, the group $K_1(C^*(E))$ is free. (Remarkably, this is the only restriction on the K-theory.)

The Kernel and Cokernel of a Finite Matrix

Let A be an $m \times n$ matrix with integer entries, and consider $A : \mathbb{Z}^n \to \mathbb{Z}^m$ by left multiplication. By performing elementary row and column operations (over $\mathbb{Z}$) to A we obtain

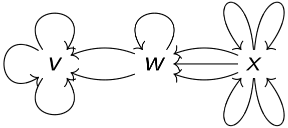
$$D = \begin{pmatrix} d_1 & & & & \cdots & 0 \\ & \ddots & & & & \vdots \\ & & d_k & & & \\ & & & 0 & & \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix}$$

where $d_1, \ldots, d_k$ are nonzero integers with $k \leq \min\{m, n\}$. Then

$$\operatorname{coker} A \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m-k}$$

$$\ker A \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k}.$$

Let E be the graph



Then E is row-finite with no sinks, and

$$A_E = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix} \qquad \text{and} \qquad A_E^t - I = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

One can perform elementary row and column operations on $A_E^t - I$ to obtain

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

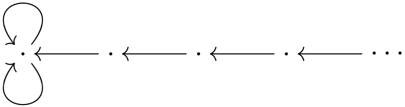
and therefore

$$K_0(C^*(E)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}$$

and

$$K_1(C^*(E)) \cong \mathbb{Z}.$$

Let E be the graph



Then E is row-finite with no sinks, and the vertex matrix of this graph is

$$A_E = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

and

$$A_E^t - I = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \\ 0 & 0 & -1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

One can show this map is injective and surjective, so

$$K_0(C^*(E)) = \operatorname{coker}(A_E^t - I) = 0$$

and

$$K_1(C^*(E)) = \ker(A_E^t - I) = 0.$$

What about when E has singular vertices (i.e., sinks or infinite emitters)?

Theorem

With respect to the decomposition $E^0 = E^0_{\text{reg}} \sqcup E^0_{\text{sing}}$, the vertex matrix of E has the form

$$A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have entries in $\mathbb{N}$ and the $*$'s have entries in $\mathbb{N} \cup \{\infty\}$. Then

$$\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_{v \in E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{v \in E^0} \mathbb{Z}$$

and

$$K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \qquad K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}.$$

This can be proven using desingularization or by direct methods.

Recall . . .

Theorem (The Dichotomy for Simple Graph Algebras)

Let E be a row-finite graph. If $C^*(E)$ is simple, then either

1. *$C^*(E)$ is an AF-algebra if E contains no cycles; or*
2. *$C^*(E)$ is purely infinite if E contains a cycle.*

The Dichotomy allows us to classify all simple graph C^*-algebras by their K-theory!

Since any simple graph C^*-algebras are either AF or purely infinite, we may use either Elliott's Theorem or the Kirchberg-Phillips Classification Theorem.

Theorem (Elliott's Theorem)

Let A and B be AF-algebras. Then A and B are Morita equivalent if and only if $(K_0(A), K_0(A)^+) \cong (K_0(B), K_0(B)^+)$. *That is, the ordered* K_0*-group of an AF-algebra is a complete Morita equivalence invariant.*

If A and B are both unital, then A and B are isomorphic if and only if $(K_0(A), K_0(A)^+, [1_A]_0) \cong (K_0(B), K_0(B)^+, [1_B]_0)$.

Theorem (Kirchberg-Phillips)

Let A and B be purely infinite, simple, separable, nuclear C^*-algebras that satisfy the Universal Coefficients Theorem.

1. *If A and B are both unital, then A is isomorphic to B if and only if $(K_0(A), [1]_0) \cong (K_0(B), [1]_0)$ and $K_1(A) \cong K_1(B)$. Furthermore, A is Morita equivalent to B if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.*

2. *If A and B are nonunital, then A is isomorphic to B if and only if A is Morita equivalent to B if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.*

Thus for any purely infinite, simple, separable, nuclear C^*-algebra A that satisfies the Universal Coefficients Theorem, $(K_0(A), K_1(A))$ is a complete Morita equivalence invariant.

Note: Graph C^*-algebras are separable, nuclear, and satisfy the UCT.

Any simple graph C^*-algebra is either AF or purely infinite.

Corollary

If A is a simple graph C^*-algebra, then

$$((K_0(A), K_0(A)^+), K_1(A))$$

is a complete Morita equivalence invariant for A.

What is the range of this invariant for simple graph C^*-algebras?

For AF-algebras the range of the invariant $(K_0(A), K_0(A)^+)$ is the collection of all Riesz groups; i.e. direct limits of the form $\varinjlim(\mathbb{Z}^n, (\mathbb{Z}^+)^n)$.

Theorem (Drinen)

If A is an AF-algebra, then there exists a row-finite graph E such that $C^*(E)$ is Morita equivalent to A.

Thus the AF graph C^*-algebras (i.e., the C^*-algebras of graphs with no cycles) have all possible Riesz groups as their K-theories.

For simple AF graph C^*-algebras, we obtain the collection of all simple Riesz groups.

For purely infinite simple separable nuclear C^*-algebras, all pairs of countable abelian groups are possible as the K-theory groups.

For graph C^*-algebras, we know that $K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$, so $K_1(C^*(E))$ is a free group. This is the only obstruction.

Theorem (Szymański)

Let (G_0, G_1) be any pair of countable abelian groups with G_1 free. Then there exists a row-finite transitive graph E such that $K_0(C^*(E)) \cong G_0$ and $K_1(C^*(E)) \cong G_1$.

This shows any Kirchberg algebra with free K_1-group is Morita equivalent to a graph C^*-algebra.

Thus the range of the invariant for purely infinite simple graph C^*-algebras is all pairs of countable abelian groups (G_0, G_1) with G_1 free.

In addition to K-theory, the Ext group and the the K-homology of graph C^*-algebras has been computed. If E is a graph, then with respect to the decomposition $E^0 = E^0_{\text{reg}} \sqcup E^0_{\text{sing}}$, the vertex matrix of E has the form

$$A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have entries in $\mathbb{N}$ and the $*$'s have entries in $\mathbb{N} \cup \{\infty\}$. Then

$$\begin{pmatrix} B - I & C \end{pmatrix} : \prod_{v \in E^0} \mathbb{Z} \to \prod_{v \in E^0_{\text{reg}}} \mathbb{Z}.$$

Theorem (T)

$\operatorname{Ext}(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B - I & C \end{pmatrix}$

Theorem (Yi)

$K^0(C^*(E)) \cong \ker \begin{pmatrix} B - I & C \end{pmatrix}$
$K^1(C^*(E)) \cong \operatorname{Ext}(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B - I & C \end{pmatrix}$

Since we frequently want to know about Morita equivalence, we are often concerned with stability of graph C^*-algebras.

Recall: A C^*-algebra A is *stable* if $A \cong A \otimes \mathcal{K}$. We call $A \otimes \mathcal{K}$ the *stabillization* of A.

Fact: If A and B are separable C^*-algebras, then A is Morita equivalent to B if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.

Definition

If E is a graph, then a *graph trace* on E is a function $g : E^0 \to \mathbb{R}^+$ with the following two properties:

1. For any $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$ we have
$$g(v) = \sum_{s(e)=v} g(r(e)).$$
2. For any infinite emitter $v \in G^0$ and any finite set of edges $e_1, \ldots, e_n \in s^{-1}(v)$ we have
$$g(v) \geq \sum_{i=1}^{n} g(r(e_i)).$$

Theorem (T)

If E is a graph, then the following are equivalent.

(a) $C^*(E)$ *is stable*

(b) $C^*(E)$ *has no nonzero unital quotients and no tracial states*

(c) *Every vertex in E that is on a cycle may be reached by an infinite number of other vertices and there are no graph traces on E.*

Definition

If E is a graph and $v \in E^0$ is a vertex, then by *adding a head to* v we mean attaching a graph of the form

$$\cdots \xrightarrow{e_4} v_3 \xrightarrow{e_3} v_2 \xrightarrow{e_2} v_1 \xrightarrow{e_1} v$$

to E.

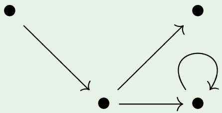
Theorem

If E is a graph, let $\tilde{E}$ be the graph obtained by adding a head to each vertex of E. Then $C^*(\tilde{E})$ is the stabilization of $C^*(E)$; that is,

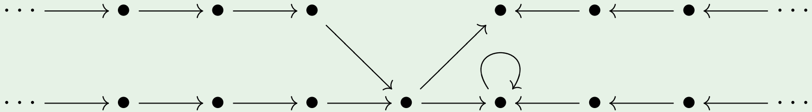
$$C^*(\tilde{E}) \cong C^*(E) \otimes \mathcal{K}.$$

Example

If E is the graph



then $\tilde{E}$ is the graph



and $C^*(\tilde{E}) \cong C^*(E) \otimes \mathcal{K}$.

Example

If E is the following graph with one vertex and infinitely many edges, then $C^*(E) \cong \mathcal{O}_\infty$

$$\bullet \circlearrowleft \infty$$

and $\tilde{E}$ is the graph

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \circlearrowleft \infty$$

so that $C^*(\tilde{E}) \cong \mathcal{O}_\infty \otimes \mathcal{K}$

Corollary

The class of graph C^*-algebras is closed under stabilization.