Talk 3: Graph C^*-algebras as Cuntz-Pimsner algebras

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July, 2010

Pimsner described a method for taking a C^*-correspondence X over a C^*-algebra A, and constructing a C^*-algebra $\mathcal{O}_X$ that generalizes the Cuntz-Krieger construction and the construction of crossed products by $\mathbb{Z}$.

$\mathcal{O}_X$ is called the Cuntz-Pimsner algebra, and the collection of these algebras compose a class of C^*-algebras that is extraordinarily rich.

Information about $\mathcal{O}_X$ is very densely codified in (X, A), and determining how to extract it has been the focus of much current effort.

Definition

If A is a C^*-algebra, then a *right Hilbert A-module* is a Banach space X together with a right action of A on X and an A-valued inner product $\langle \cdot, \cdot \rangle_A$ satisfying

(i) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$

(ii) $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$

(iii) $\langle \xi, \xi \rangle_A \geq 0$ and $\|\xi\| = \langle \xi, \xi \rangle_A^{1/2}$

for all $\xi, \eta \in X$ and $a \in A$.

$\mathcal{L}(X)$ is the C^*-algebra of adjointable operators on X

$\mathcal{K}(X)$ is the closed two-sided ideal of compact operators given by

$$\mathcal{K}(X) := \overline{\mathrm{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in X\}$$

where $\Theta_{\xi,\eta}(\zeta) := \xi \langle \eta, \zeta \rangle_A$.

Definition

If A is a C^*-algebra, then a *C^*-correspondence* is a right Hilbert A-module X together with a $*$-homomorphism $\phi : A \to \mathcal{L}(X)$. We consider ϕ as giving a left action of A on X by setting $a \cdot x := \phi(a)x$.

Definition

If X is a C^*-correspondence over A, then a *representation* of X into a C^*-algebra B is a pair (π, t) consisting of a $*$-homomorphism $\pi : A \to B$ and a linear map $t : X \to B$ satisfying

(i) $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle_A)$

(ii) $t(\phi(a)\xi) = \pi(a)t(\xi)$

(iii) $t(\xi a) = t(\xi)\pi(a)$

for all $\xi, \eta \in X$ and $a \in A$.

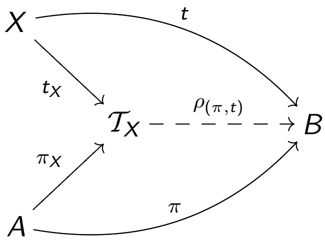
A representation (π, t) is said to be *injective* if π is injective. Note that in this case t will also be isometric since

$$\|t(\xi)\|^2 = \|t(\xi)^* t(\xi)\| = \|\pi(\langle \xi, \xi \rangle_A)\| = \|\langle \xi, \xi \rangle_A\| = \|\xi\|^2.$$

There is a C^*-algebra, denoted $\mathcal{T}_X$ and a representation (π_X, t_X) of X in $\mathcal{T}_X$ that is universal in the following sense:

- $\mathcal{T}_X$ is generated as a C^*-algebra by $\operatorname{im}\pi_X \cup \operatorname{im} t_X$

- given any representation (π, t) in a C^*-algebra B, then there is a C^*-homomorphism of $\mathcal{T}_X$ into B, denoted $\rho_{(\pi,t)}$, such that $\pi = \rho_{(\pi,t)} \circ \pi_X$ and $t = \rho_{(\pi,t)} \circ t_X$.



We call $\mathcal{T}_X$ the Toeplitz algebra associated to X.

Definition

For a representation (π, t) of a C^*-correspondence X on B there exists a $*$-homomorphism $\pi^{(1)} : \mathcal{K}(X) \to B$ with the property that

$$\pi^{(1)}(\Theta_{\xi,\eta}) = t(\xi)t(\eta)^*.$$

Moreover, if (π, t) is an injective representation, then $\pi^{(1)}$ will be injective as well.

Definition

If X is a C^*-correspondence over A and K is an ideal in $J(X)$, then we say that a representation (π, t) is *coisometric on* K, or is K*-coisometric* if

$$\pi^{(1)}(\phi(a)) = \pi(a) \qquad \text{for all } a \in K.$$

Definition

For an ideal I in a C^*-algebra A we define

$$I^\perp := \{a \in A : ab = 0 \text{ for all } b \in I\}.$$

If X is a C^*-correspondence over A, we define an ideal J_X of A by

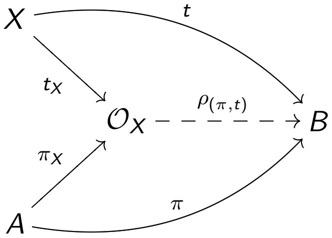
$$J_X := \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^\perp.$$

Note that $J_X = \phi^{-1}(\mathcal{K}(X))$ when ϕ is injective, and that J_X is the maximal ideal on which the restriction of ϕ is an injection into $\mathcal{K}(X)$.

There is a C^*-algebra, denoted $\mathcal{O}_X$ and a J_X-coisometric representation (π_X, t_X) of X in $\mathcal{O}_X$ that is universal in the following sense:

- $\mathcal{O}_X$ is generated as a C^*-algebra by $\operatorname{im} \pi_X \cup \operatorname{im} t_X$

- given any J_X-coisometric representation (π, t) in a C^*-algebra B, then there is a C^*-homomorphism of $\mathcal{O}_X$ into B, denoted $\rho_{(\pi,t)}$, such that $\pi = \rho_{(\pi,t)} \circ \pi_X$ and $t = \rho_{(\pi,t)} \circ t_X$.



We call $\mathcal{O}_X$ the Cuntz-Pimsner algebra associated to X. (Pimsner made the definition when ϕ is injective, Katsura in general.)

Graph C^*-algebras

Recall that if $E := (E^0, E^1, r, s)$ is a directed graph, then $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family; i.e., a collection of partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal range projections together with a collection of mutually orthogonal projections $\{p_v : v \in E^0\}$ that satisfy

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$

Example (The Graph C^*-correspondence)

If $E = (E^0, E^1, r, s)$ is a graph, we define $A := C_0(E^0)$ and

$$X(E) := \{x : E^1 \to \mathbb{C} : \text{the function } v \mapsto \sum_{\{f \in E^1 : r(f) = v\}} |x(f)|^2 \text{ is in } C_0(E^0)\}.$$

Then $X(E)$ is a C^*-correspondence over A with the operations

$$(x \cdot a)(f) := x(f)a(r(f)) \text{ for } f \in E^1$$

$$\langle x, y \rangle_A(v) := \sum_{\{f \in E^1 : r(f) = v\}} \overline{x(f)}y(f) \text{ for } f \in E^1$$

$$(\phi(a)x)(f) := a(s(f))x(f) \text{ for } f \in E^1.$$

Note that we could write $X(E) = \bigoplus_{v \in E^0}^0 \ell^2(r^{-1}(v))$ where this denotes the c_0 direct sum of the $\ell^2(r^{-1}(v))$'s. Also note that $X(E)$ and A are densely spanned by the point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$, respectively.

Suppose (π, t) is a representation of $X(E)$ into B. Let $P_v := \pi(\delta_v)$ and $S_e := t(\delta_e)$.

Then $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle_A)$ shows that

$$S_e^* S_e = t(\delta_e)^* t(\delta_e) = \pi(\langle \delta_e, \delta_e \rangle) = \pi(\delta_{r(e)}) = P_{r(e)}.$$

and $t(\phi(a)\xi) = \pi(a)t(\xi)$ shows that

$$P_{s(e)} S_e = \pi(\delta_{s(e)}) t(\delta_e) = t(\phi(\delta_{s(e)}\delta_e) = t(\delta_e) = S_e$$

so $S_e S_e^* \leq P_{s(e)}$.

Thus two of the Cuntz-Krieger relations follow from the representation properties.

Since $J_{X(E)}$ is an ideal of $C_0(E^0)$, it has the form $\overline{\operatorname{span}}\{\delta_v : v \in S\}$ for some $S \subseteq E^0$. In fact,

$$J_{X(E)} = \overline{\operatorname{span}}\{\delta_v : v \in E^0_{\text{reg}}\}.$$

When $v \in E^0_{\text{reg}}$, a short calculation shows

$$\phi(\delta_v) = \sum_{s(e)=v} \Theta_{\delta_e,\delta_e}.$$

Thus if (π, t) is coisometric on J_X, for any $v \in E^0_{\text{reg}}$ we have

$$\begin{aligned} P_v = \pi(\delta_v) = \pi^{(1)}(\phi(\delta_v)) &= \pi^{(1)}\left(\sum_{s(e)=v} \Theta_{\delta_e,\delta_e}\right) \\ &= \sum_{s(e)=v} t(\delta_e)t(\delta_e)^* = \sum_{s(e)=v} S_e S_e^*. \end{aligned}$$

so $\{S_e, P_v\}$ is a Cuntz-Krieger E-family.

Let (π_X, t_X) be a universal J_X-coisometric representation of X into $\mathcal{O}_{X(E)}$. Then $\{\pi_X(\delta_v), t_X(\delta_e)\}$ is a Cuntz-Krieger E-family generating $\mathcal{O}_{X(E)}$.

Moreover, this Cuntz-Krieger E-family is universal. If $\{S_e, P_v\}$ is a Cuntz-Krieger E-family in a C^*-algebra B, then we may define $\pi : C_0(E^0) \to B$ by

$$\pi(a) = \sum_{v \in E^0} a(v) P_v,$$

and $t : X(E) \to B$ by

$$t(x) = \sum_{e \in E^1} x(e) S_e.$$

Then (π, t) is a J_X-coisometric representation of X into B. Thus there exists a $*$-homomorphism $\rho : \mathcal{O}_{X(E)} \to B$ such that $\rho \circ \pi_X = \pi$ and $\rho \circ t_X = t$. Hence $\rho(\pi_X(\delta_v)) = P_v$ and $\rho(t_X(\delta_e)) = S_e$.

Hence $\{\pi_X(\delta_v), t_X(\delta_e)\}$ is a *universal* Cuntz-Krieger E-family generating $\mathcal{O}_{X(E)}$. Thus

$$\mathcal{O}_{X(E)} \cong C^*(E).$$

Properties of the graph and properties of the graph correspondence are related.

Property of X(E)	**Property of E**
$\phi(\delta_v) \in \mathcal{K}(X(E))$	v emits a finite number of edges
$\operatorname{im}\phi \subseteq \mathcal{K}(X(E))$	E is row-finite
$\delta_v \in \ker\phi$	v is a sink
ϕ is injective	E has no sinks
$\{\langle x, y\rangle_A : x, y \in X(E)\}$ is dense in A	E has no sources

Note: Row-finite with no sinks corresponds to $\phi(A) \subseteq \mathcal{K}(X(E))$ with ϕ injective.

We can now seek to describe versions of graph C^*-algebra theorems for general Cuntz-Pimsner algebras.

We'll start with the gauge action . . .

If $\mathcal{O}_X$ is a Cuntz-Pimsner algebra associated to a C^*-correspondence X, and if (π_X, t_X) is a universal J_X-coisometric representation, then for any $z \in \mathbb{T}$ we have that (π_X, zt_X) is also a universal J_X-coisometric representation.

Hence by the universal property, there exists a homomorphism $\gamma_z : \mathcal{O}_X \to \mathcal{O}_X$ such that $\gamma_z(\pi_X(a)) = \pi_X(a)$ for all $a \in A$ and $\gamma_z(t_X(\xi)) = zt_X(\xi)$ for all $\xi \in X$. Since $\gamma_{z^{-1}}$ is an inverse for this homomorphism, we see that γ_z is an automorphism. Thus we have an action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_X$ with the property that $\gamma_z(\pi_X(a)) = \pi_X(a)$ and $\gamma_z(t_X(\xi)) = zt_X(\xi)$.

Theorem (Gauge-Invariant Uniqueness)

Let X be a C^*-correspondence over A, and let $\rho : \mathcal{O}_X \to B$ a $*$-homomorphism between C^*-algebras with the property that $\rho|_{\operatorname{im} \pi_X}$ is injective. If there exists a gauge action β of $\mathbb{T}$ on B such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$, then ρ is injective.

Note: There is no Cuntz-Krieger Uniqueness Theorem for Cuntz-Pimsner algebras, because there is no satisfactory notion of Condition (L) for C^*-correspondences.

Recall that if E is a row-finite graph, gauge-invariant ideals in $C^*(E)$ correspond to saturated hereditary subsets in E^0.

In the graph correspondence

$$(x \cdot a)(f) := x(f)a(r(f)) \qquad \text{and} \qquad (\phi(a)x)(f) := a(s(f))x(f)$$

Subsets of E^0 correspond to ideals in $C_0(E^0)$ by $H \leftrightarrow \overline{\mathrm{span}}\{\delta_v : v \in H\}$.

Definition

Let X be a C^*-correspondence over A. We say that an ideal $I \triangleleft A$ is X-*invariant* if $\phi(I)X \subseteq XI$. We say that an X-invariant ideal $I \triangleleft A$ is X-*saturated* if

$$a \in J_X \text{ and } \phi(a)X \subseteq XI \implies a \in I.$$

H is hereditary $\iff \overline{\mathrm{span}}\{\delta_v : v \in H\}$ is $X(E)$-invariant
H is saturated $\iff \overline{\mathrm{span}}\{\delta_v : v \in H\}$ is $X(E)$-saturated.

Theorem

Let X be a C^*-correspondence with the property that $\operatorname{im} \phi_X \subseteq \mathcal{K}(X)$ and ϕ is injective. Also let (π_X, t_X) be a universal J_X-coisometric representation of X into $\mathcal{O}_X$. Then there is a lattice isomorphism from the X-saturated X-invariant ideals of A onto the gauge-invariant ideals of $\mathcal{O}_X$ given by

$$I \mapsto \mathcal{I}(I) := \text{the ideal in } \mathcal{O}_X \text{ generated by } \pi_X(I).$$

Furthermore, $\mathcal{O}_X/\mathcal{I}(I) \cong \mathcal{O}_{X/XI}$, and the ideal $\mathcal{I}(I)$ is Morita equivalent to $\mathcal{O}_{XI}$.

In general, gauge-invariant ideals of $\mathcal{O}_X$ correspond to pairs of ideals coming from A (the so-called O-pairs of Katsura), which generalize the admissible pairs (H, S) of saturated hereditary subsets and breaking vertices.

Some other facts:

The dichotomy does not hold: there are simple Cuntz-Pimsner algebras that are neither AF nor purely infinite.

In addition, a six-term exact sequence for the K-groups of $\mathcal{O}_X$ has been established that allows one to calculate the K-theory of $\mathcal{O}_X$ in certain situations.

$$\begin{array}{ccccc} K_0(J_X) & \longrightarrow & K_0(A) & \longrightarrow & K_0(\mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_X) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J_X) \end{array}$$

All possible K-groups can be realized as the K-theory of Cuntz-Pimsner algebras.

Consider the graph C^*-correspondence case:

$$\begin{array}{ccccc} K_0(J_X) & \longrightarrow & K_0(A) & \longrightarrow & K_0(\mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_X) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J_X) \end{array}$$

We have $A = C_0(E^0)$ and $J_X = \overline{\operatorname{span}}\{\delta_v : v \in E^0_{\text{reg}}\}$. Since these are spaces of continuous functions on discrete spaces, the K_1 groups are zero, and $K_0(A) \cong \bigoplus_{v \in E^0} \mathbb{Z}$ and $K_0(J_X) \cong \bigoplus_{v \in E^0_{\text{reg}}} \mathbb{Z}$. Thus the exact sequence becomes

$$0 \longrightarrow K_1(\mathcal{O}_X) \longrightarrow \bigoplus_{v \in E^0_{\text{reg}}} \mathbb{Z} \xrightarrow{\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}} \bigoplus_{v \in E^0} \mathbb{Z} \longrightarrow K_0(\mathcal{O}_X) \longrightarrow 0$$

where $A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$, and we recover the graph C^*-algebra results

$$K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \qquad \text{and} \qquad K_1(C^*(E)) \cong \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}.$$

We can also generalize certain constructions from the graph C^*-algebra case to general Cuntz-Pimsner algebras.

Let's consider the construction of "adding tails to sinks".

Let E be a graph and $v \in E^0$ be a sink. We add a tail to E to form a graph F by attaching

$$v \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

and then $C^*(E)$ is isomorphic to a full corner of $C^*(F)$ determined by the projection $p := \sum_{v \in E^0} p_v$.

$$v \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

What happens at the C^*-correspondence level?

$$F^1 = E^1 \cup \{e_1, e_2, e_3, \ldots\} \qquad \text{and} \qquad F^0 = E^0 \cup \{v_1, v_2, v_3, \ldots\}$$

$$X(F) = X(E) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C} = X(E) \oplus C_0(\{e_1, e_2, e_3, \ldots\})$$

$$C_0(F^0) = C_0(E^0) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C} = C_0(E^0) \oplus C_0(\{v_1, v_2, v_3, \ldots\}).$$

Recall:
$(x \cdot a)(f) := x(f)a(r(f))$
$\langle x, y\rangle(v) = \sum_{r(f)=v} \overline{x(f)}y(f)$
$(\phi(a)x)(f) := a(s(f))x(f)$

So the right action and inner product are the usual ones given to the direct sum $X(E) \oplus C_0(\{e_1, e_2, e_3, \ldots\})$ over $C_0(E^0) \oplus C_0(\{v_1, v_2, v_3, \ldots\})$, but the left action "shifts things one entry to the right". Also recall v is a sink iff $\delta_v \in \ker \phi$.

ADDING TAILS TO GENERAL CORRESPONDENCES

Let X be a C^*-correspondence over A with left action $\phi : A \to \mathcal{L}(X)$. Define the *tail of* X to be the c_0-direct sum $T := \bigoplus_{i=1}^{\infty} \ker \phi$.

Form a new C^*-correspondence $Y := X \oplus T$ over $B := A \oplus T$ with

$$(\xi, (f_1, f_2, \ldots)) \cdot (a, (g_1, g_2, \ldots)) := (\xi \cdot a, (f_1 g_1, f_2 g_2, \ldots))$$

the inner product is given by

$$\langle (\xi, (f_1, f_2, \ldots)), (\nu, (g_1, g_2, \ldots)) \rangle_B := (\langle \xi, \nu \rangle_A, (f_1^* g_1, f_2^* f_2, \ldots))$$

and left action $\phi_B : B \to \mathcal{L}(Y)$ is

$$\phi_B(a, (f_1, f_2, \ldots))(\xi, (g_1, g_2, \ldots)) := (\phi(a)(\xi), (a g_1, f_1 g_2, f_2 g_3, \ldots))$$

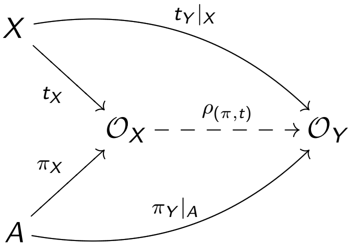
Note: The left action ϕ_B on Y is injective. Thus $\mathcal{O}_Y$ is the algebra defined by Pimsner (Katsura's definition not needed).

Add a tail to X to obtain a correspondence $Y := X \oplus T$ over $B := A \oplus T$.

Let (π_Y, t_Y) be a universal J_Y-coisometric representation of Y. Then $(\pi, t) := (\pi_Y|_A, t_Y|_X)$ is a J_X-coisometric representation of X in $\mathcal{O}_Y$. Furthermore, $\rho_{(\pi,t)} : \mathcal{O}_X \to C^*(\pi_X, t_X) \subseteq \mathcal{O}_Y$ is an isomorphism onto the C^*-subalgebra of $\mathcal{O}_Y$ generated by

$$\{\pi_Y(a, \vec{0}), t_Y(\xi, \vec{0}) : a \in A \text{ and } \xi \in X\}$$

and this C^*-subalgebra is a full corner of $\mathcal{O}_Y$.



Consequently, $\mathcal{O}_X$ is naturally isomorphic to a full corner of $\mathcal{O}_Y$.

This result often allows one to restrict to the case when ϕ is injective and then extend using Morita equivalence.

In particular, we can sometimes extend results proven for Pimsner's algebras to the algebras more generally defined by Katsura.

Example: Fowler, Muhly, and Raeburn proved the Gauge-Invariant Uniqueness Theorem for Cuntz-Pimsner algebras of C^*-correspondences with ϕ injective.

Using the method of adding tails, we can extend the Gauge-Invariant Uniqueness Theorem to Cuntz-Pimsner algebras of general C^*-correspondences.

Another Example: If we let X be a C^*-correspondence, and let (π_X, t_X) be a universal representation into $\mathcal{T}_X$, then the *tensor algebra* $\mathcal{T}_X^+$ is defined to be the norm-closed algebra generated by $\operatorname{im} \pi_X \cup \operatorname{im} t_X$.

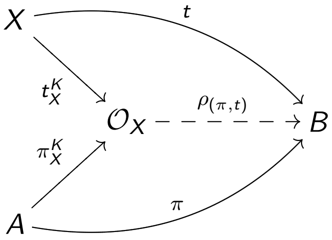
Muhly and Solel showed that if ϕ is injective, then the C^*-envelope of $\mathcal{T}_X^+$ is $\mathcal{O}_X$.

Katsoulis and Kribs showed that by adding tails one can extend the Muhly-Solel result, and prove that, in general, the the C^*-envelope of $\mathcal{T}_X^+$ is $\mathcal{O}_X$.

RELATIVE CUNTZ-PIMNSER ALGEBRAS

If X is a C^*-correspondence over A, and K is an ideal in A with $K \subseteq J_X$, then we may define the *relative Cuntz-Pimsner algebra* $\mathcal{O}(K, X)$ to be the C^*-algebra generated by a universal K-coisometric representation.

In other words, there exists a K-coisomteric representation (π_X^K, t_X^K) of X into $\mathcal{O}(K, X)$ such that $\mathcal{O}(K, X)$ is generated by $\operatorname{im} \pi_X^K \cup \operatorname{im} t_X^K$, and whenever (π, t) is a K-coisometric representation of X into a C^*-algebra B, then there exists a $*$-homomorphism $\rho_{(\pi,t)} : \mathcal{O}(K, X) \to B$ making the following diagram commute.



Note: $\mathcal{O}(J_X, X) = \mathcal{O}_X$ and $\mathcal{O}(\{0\}, X) = \mathcal{T}_X$.

In the graph setting, $K \subseteq J_{X(E)}$ implies $K = \overline{\mathrm{span}}\{\delta_v : v \in V\}$ for some $V \subseteq E^0_{\mathrm{reg}}$.

The relative Cuntz-Pimsner algebra is a *relative graph C^*-algebra*, $C^*(E, V)$, which is the universal C^*-algebra generated by a collection of partial isometries $\{s_e : e \in E^1\}$ with commuting range projections together with a collection of mutually orthogonal projections $\{p_v : v \in E^0\}$ that satisfy

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in V$

The relative graph C^*-algebras and relative Cuntz-Pimnser algebras arise naturally when describing subalgebras of graph algebras and when describing quotients of graph algebras.

Katsura has shown that every relative Cuntz-Pimsner algebra is a Cuntz-Pimsner algebra; i.e., given a relative Cuntz-Pimsner algebra $\mathcal{O}(K, X)$ there exists a C^*-correspondence Y such that $\mathcal{O}(K, X) \cong \mathcal{O}_Y$.

For relative graph C^*-algebra $C^*(E, V)$, there exists a graph F such that $C^*(E, V) \cong C^*(F)$. We may obtain F by "splitting"' vertices in $E^0_{\mathrm{reg}} \setminus V$.