Talk 4: Classification of nonsimple graph C^*-algebras

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We have seen we can compute K-theory for graph C^*-algebras. Also, we can use Elliott's Theorem and the Kirchberg-Phillips Classification Theorem to classify simple graph C^*-algebras up to Morita equivalence (i.e., stable isomorphism).

What about in the nonsimple case?

We will consider the 1-ideal case first.

Graph C^*-algebras provide a good testing grounds for conjectures and preliminary theories — they are simultaneously rich and tractable. All simple graph C^*-algebras are AF or purely infinite, so graph C^*-algebras with one ideal will be extensions of these two types. Thus we can have "mixing" of different types, but the "mixing" is not as complicated as with general C^*-algebras.

C*(E)	Properties of E
Unital	finite number of vertices
Finite Dim.	finite graph with no cycles
Simple	(1) Every cycle has an exit (2) No saturated hereditary sets
Simple and Purely Infinite	(1) Every cycle has an exit (2) No saturated hereditary sets (3) Every vertex can reach a cycle
AF	no cycles

K-theory for Graph C^*-algebras

Remark

Let $E = (E^0, E^1, r, s)$ be a graph with no singular vertices, and let A_E be the $E^0 \times E^0$ matrix $A_E(v, w) := \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}$. Then $A_E^t - I : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$ and

$$K_0(C^*(E)) \cong \operatorname{coker}(A_E^t - I) \qquad K_1(C^*(E)) \cong \ker(A_E^t - I)$$

Remark

If E has singular vertices, in the decomposition $E^0 = E^0_{\text{reg}} \cup E^0_{\text{sing}}$ we have

$$A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \mathbb{Z}^{E^0_{\text{reg}}} \to \mathbb{Z}^{E^0}.$$

Then

$$K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \qquad K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}.$$

Classification up to stable isomorphism

Theorem (Elliott)

Let A and B be AF C^*-algebras. Then $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ if and only if

$$(K_0(A), K_0(A)^+) \cong (K_0(B), K_0(B)^+)$$

Theorem (Kirchberg-Phillips)

Let A and B be Kirchberg algebras (purely infinite, simple, separable, nuclear, and in the UCT class). Then $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.

Any simple graph C^*-algebra is either AF (and classified by Elliott's Theorem) or purely infinite (and classified by the Kirchberg-Phillips Theorem).

What about classification in the nonsimple case?

Elliott's theorem holds for nonsimple AF-algebras. (Recall: ideals and quotients of AF-algebras are AF).

Meyer and Nest have proven that certain purely infinite C^*-algebras are classified by their filtrated K-theory. (All ideals and quotients are purely infinite.)

Restorff has proven that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified by their filtrated K-theory. (All ideals and quotients are purely infinite.)

Graph C^*-algebras may have a mixture: ideals and quotients can be either AF or purely infinite.

Let A be a graph C^*-algebra with a unique proper nontrivial ideal I.

$$e: \qquad 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

There are four cases:

Case	I	A/I
[11]	AF	AF
$[1\infty]$	AF	Kirchberg
$[\infty 1]$	Kirchberg	AF
$[\infty\infty]$	Kirchberg	Kirchberg

There is a result of Eilers, Restorff, and Ruiz that deals with these mixed cases.

The Invariant

For an extension

$$e: \qquad 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

we let $K_{\mathrm{six}}(e)$ denote the cyclic six-term exact sequence of K-groups

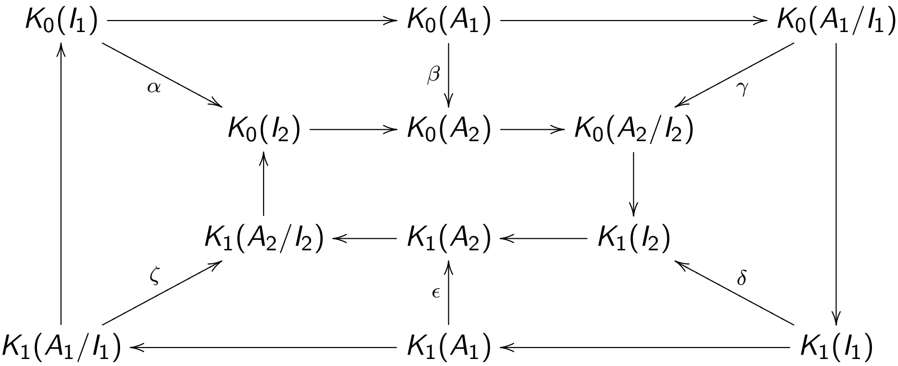
$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

where $K_0(I)$, $K_0(A)$, and $K_0(A/I)$ are viewed as ordered groups.

$$e_1: \quad 0 \longrightarrow I_1 \longrightarrow A_1 \longrightarrow A_1/I_1 \longrightarrow 0$$

$$e_2: \quad 0 \longrightarrow I_2 \longrightarrow A_2 \longrightarrow A_2/I_2 \longrightarrow 0$$

$K_{\text{six}}(\text{Ext}_1) \cong K_{\text{six}}(\text{Ext}_2)$ if $\exists$ isomorphisms α, β, γ, δ, ϵ, and ζ with



and where α, β, and γ are isomorphisms of ordered groups.

Definition

We will be interested in classes $\mathcal{C}$ of separable nuclear unital simple C^*-algebras in the bootstrap category $\mathcal{N}$ satisfying the following properties:

(I) Any element of $\mathcal{C}$ is either purely infinite or stably finite.

(II) $\mathcal{C}$ is closed under tensoring with M_n, where M_n is the C^*-algebra of n by n matrices over $\mathbb{C}$.

(III) If A is in $\mathcal{C}$, then any unital hereditary C^*-subalgebra of A is in $\mathcal{C}$.

(IV) For all A and B in $\mathcal{C}$ and for all x in $KK(A, B)$ which induce an isomorphism from $(K_*^+(A), [1_A])$ to $(K_*^+(A), [1_B])$, there exists a $*$-isomorphism $\alpha : A \to B$ such that $KK(\alpha) = x$.

Definition

If B is a separable stable C^*-algebra, then we say that B has the *corona factorization property* if every full projection in $\mathcal{M}(B)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$.

Theorem (Eilers, Restorff, Ruiz)

Let $\mathcal{C}_I$ and $\mathcal{C}_Q$ be classes of unital nuclear separable simple C^*-algebras in the bootstrap category $\mathcal{N}$ satisfying the Properties (I)–(IV) above. Let A_1 and A_2 be in $\mathcal{C}_Q$ and let B_1 and B_2 be in $\mathcal{C}_I$ with $B_1 \otimes \mathcal{K}$ and $B_2 \otimes \mathcal{K}$ satisfying the corona factorization property. Let

$$e_1 : \quad 0 \longrightarrow B_1 \otimes \mathcal{K} \longrightarrow E_1 \longrightarrow A_1 \longrightarrow 0$$

$$e_2 : \quad 0 \longrightarrow B_2 \otimes \mathcal{K} \longrightarrow E_2 \longrightarrow A_2 \longrightarrow 0$$

be **essential** *extensions with E_1 and E_2 unital. If $K_{six}(e_1) \cong K_{six}(e_2)$, then $E_1 \otimes \mathcal{K} \cong E_2 \otimes \mathcal{K}$.*

Note: This does not apply immediately to graph C^*-algebras. Graph C^*-algebras need not be unital, and their ideals need not be stable.

But with some work, we can prove the following:

Theorem (Eilers and T)

If A is a graph C^*-algebra with exactly one proper nontrivial ideal I, then A classified up to stable isomorphism by the six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

with all K_0-groups considered as ordered groups. In other words, if

$$e_1 : \quad 0 \longrightarrow I \longrightarrow C^*(E) \longrightarrow C^*(E)/I \longrightarrow 0$$

$$e_2 : \quad 0 \longrightarrow I' \longrightarrow C^*(F) \longrightarrow C^*(F)/I' \longrightarrow 0$$

then $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ if and only if $K_{six}(e_1) \cong K_{six}(e_2)$.

Sketch of Proof

Let $\mathcal{C}_I = \mathcal{C}_Q =$ union of simple AF-algebras and Kirchberg algebras

Cases: $[1\infty], [\infty 1]$, and $[\infty\infty]$.
Given

$$e: \qquad 0 \longrightarrow I \longrightarrow C^*(E) \longrightarrow C^*(E)/I \longrightarrow 0$$

we prove there exists a full projection $p \in C^*(E)$ such that pIp is stable.
Since p is full, the vertical maps in

$$\begin{array}{cccccccccc} e' : & 0 & \longrightarrow & pIp & \longrightarrow & pAp & \longrightarrow & pAp/pIp & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ e : & 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \end{array}$$

are full inclusions and $K_{\text{six}}(e) \cong K_{\text{six}}(e')$. Since pIp is stable, $pIp = B \otimes \mathcal{K}$ for some B in our class. Thus e' has the appropriate form.

Case: $[11]$. Apply Elliott's Theorem.

In fact, we can prove slightly more in the $[1\infty]$ case.

Theorem

If A is a the C^*-algebra of a graph satisfying Condition (K), and if A has a largest proper ideal I such that I is an AF-algebra, then A is classified up to stable isomorphism by the six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

with $K_0(I)$ considered as an ordered group.

Examples

Consider the graph with one vertex

E

and n edges.

Cases:

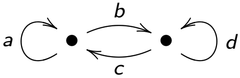
$n = 0$	$C^*(E) \cong \mathbb{C}$
$n = 1$	$C^*(E) \cong C(\mathbb{T})$
$1 < n < \infty$	$C^*(E) \cong \mathcal{O}_n$
$n = \infty$	$C^*(E) \cong \mathcal{O}_\infty$

Stable isomorphism class determined uniquely by n.

Examples

Consider the graph with two vertices

E

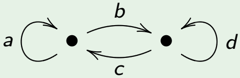


with $A_E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (We'll assume this graph satisfies Condition (K).)

We need to consider 5 cases and subcases.

Case I: $b \neq 0$ and $c \neq 0$

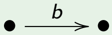
E



In this case $C^*(E)$ is simple and purely infinite and stable isomorphism class is determined by $K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$.

Case II: $a = 0$, $1 \leq b < \infty$, $c = 0$, $d = 0$,

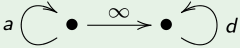
E



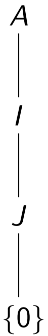
In this case $C^*(E)$ is simple and finite dimensional, and $C^*(E) \cong M_{b+1}(\mathbb{C})$.

So each value of b gives a different isomorphism class, but they are all stably isomorphic.

Case III: $1 < a < \infty$, $b = \infty$, $c = 0$

E 

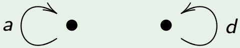
In this case $C^*(E)$ has ideal lattice



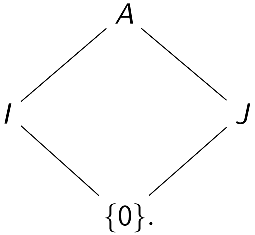
and the stable isomorphism class is determined uniquely by a and d.

Case IV: $b = 0$ and $c = 0$

E



In this case $C^*(E)$ has ideal lattice



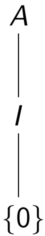
and $C^*(E) \cong \mathcal{O}_a \oplus \mathcal{O}_d$. The stable isomorphism class is determined uniquely by a and d.

Case V: E has one of the following forms

$0 < a \leq \infty$

$0 < b < \infty$

In this case $C^*(E)$ has ideal lattice



and by our theorem the stable isomorphism class is determined by $K_{\text{six}}(e)$.

a	d	b	$K_0(I) \to K_0(C^*(E)) \to K_0(C^*(E)/I)$	Case
0	0	∞	$\mathbb{Z}_{++} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_{++}$	$[\mathbf{11}]$
0	n	∞	$\mathbb{Z}_{d-1} \to \mathbb{Z}_{d-1} \oplus \mathbb{Z} \to \mathbb{Z}_{++}$	$[\infty\mathbf{1}]$
0	∞	∞	$\mathbb{Z}_{\pm} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_{++}$	$[\infty\mathbf{1}]$
n	0	n	$\mathbb{Z}_{++} \to \mathrm{coker}(\left[\begin{smallmatrix} b \\ a-1 \end{smallmatrix}\right]) \to \mathbb{Z}_{a-1}$	$[\mathbf{1}\infty]$
n	n	n	$\mathbb{Z}_{d-1} \to \mathrm{coker}(\left[\begin{smallmatrix} d-1 & b \\ 0 & a-1 \end{smallmatrix}\right]) \to \mathbb{Z}_{a-1}$	$[\infty\infty]$
n	∞	n	$\mathbb{Z}_{\pm} \to \mathrm{coker}(\left[\begin{smallmatrix} b \\ a-1 \end{smallmatrix}\right]) \to \mathbb{Z}_{a-1}$	$[\infty\infty]$
∞	0	n, ∞	$\mathbb{Z}_{++} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_{\pm}$	$[\mathbf{1}\infty]$
∞	n	n, ∞	$\mathbb{Z}_{d-1} \to \mathbb{Z}_{d-1} \oplus \mathbb{Z} \to \mathbb{Z}_{\pm}$	$[\infty\infty]$
∞	∞	n, ∞	$\mathbb{Z}_{\pm} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_{\pm}$	$[\infty\infty]$

"$\mathbb{Z}_{++}$" means $\mathbb{Z}$ with $\mathbb{Z}_+ = \mathbb{N}$ and "$\mathbb{Z}_\pm$" means $\mathbb{Z}$ ordered with $\mathbb{Z}_+ = \mathbb{Z}$.

We see that there are *several* stable isomorphism classes of C^*-algebras of graphs with two vertices.

In these cases there are many ideal lattices exhibited.

Also there are many examples of the four types $[\mathbf{11}]$, $[\mathbf{1}\infty]$, $[\infty\mathbf{1}]$, and $[\infty\infty]$.

This illustrates the need for our theorem in the classification of even basic examples of graph C^*-algebras, and it also illustrates the richness of graph C^*-algebras.

How do we calculate the invariant?

We know how to compute the K-groups, but remember that the homomorphisms are also part of the invariant.

Theorem (Carlsen, Eilers, and T)

Let E be a row-finite graph with no sinks such that $C^*(E)$ has a unique proper, nontrivial ideal I_H corresponding to a saturated hereditary subset H. The with respect to $E^0 = (E^0 \setminus H) \sqcup H$, we have $A_E = \begin{pmatrix} B & X \\ 0 & C \end{pmatrix}$. Also the six-term exact sequence

$$\begin{array}{ccccc} K_0(I_H) & \longrightarrow & K_0(C^*(E)) & \longrightarrow & K_0(C^*(E)/I_H) \\ \uparrow & & & & \downarrow \\ K_1(C^*(E)/I_H) & \longleftarrow & K_1(C^*(E)) & \longleftarrow & K_1(I_H) \end{array}$$

is isomorphic to

$$\begin{array}{ccccc} \operatorname{coker}(C^t - I) & \xrightarrow{[x]\mapsto\left[\begin{smallmatrix}0\\x\end{smallmatrix}\right]} & \operatorname{coker}\begin{pmatrix} B^t-I & 0 \\ X^t & C^t-I \end{pmatrix} & \xrightarrow{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\mapsto[a]} & \operatorname{coker}(B^t - I) \\ {\scriptstyle [X^t a] \leftarrow\!\shortmid [a]} \uparrow & & & & \downarrow {\scriptstyle 0} \\ \ker(B^t - I) & \xleftarrow{a \,\leftarrow\!\shortmid \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)} & \ker\begin{pmatrix} B^t-I & 0 \\ X^t & C^t-I \end{pmatrix} & \xleftarrow{\left(\begin{smallmatrix}0\\x\end{smallmatrix}\right) \leftarrow\!\shortmid x} & \ker(C^t - I) \end{array}$$

Open Question: What is the range of this invariant?

For simple AF pieces, we know all simple Riesz groups are possible for K_0 and we must have K_1 zero.

For simple purely infinite pieces all K_0 are possible and all free K_1 are possible.

The descending connecting map must be zero.

$$\begin{array}{ccccc}
K_0(I_H) & \longrightarrow & K_0(C^*(E)) & \longrightarrow & K_0(C^*(E)/I_H) \\
\uparrow & & & & \downarrow {\scriptstyle 0} \\
K_1(C^*(E)/I_H) & \longleftarrow & K_1(C^*(E)) & \longleftarrow & K_1(I_H)
\end{array}$$

Are there any other obstructions?

If we knew the range of the invariant, we could consider the question:

When is an extension of two simple graph C^*-algebras a graph C^*-algebra?

Note: Graph C^*-algebras are not closed under extensions.