Talk 6: Leavitt path algebras

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In its beginnings, the study of Operator Algebras received a great deal of motivation from Ring Theory.

Nowadays, both Operator Algebra and Ring Theory have great potential to influence, inspire, and explain much of the work being done in the other subject.

Today I would like to tell you about a novel way in which this interplay between Operator Algebra and Ring Theory continues.

Leavitt path algebras and graph C^*-algebras: A Brief History

It is likely that when you first learned of rings you studied the examples

$$\mathbb{Z}, \quad \text{fields}, \quad \text{matrix rings}, \quad \text{polynomial rings}$$

These all have the *Invariant Basis Number* property:

$$R^m \cong R^n \text{ (as left } R\text{-modules) implies } m = n.$$

In the 1950's Bill Leavitt showed that if R is a unital ring, then $R^1 \cong R^n$ for $n > 1$ if and only if there exist $x_1, \ldots, x_n, x_1^*, \ldots, x_n^* \in R$ such that

1. $x_i^* x_j = \delta_{ij} 1_R$
2. $\sum_{i=1}^{n} x_i x_i^* = 1_R$.

For a given field K, we let $L_K(1, n)$ denote the K-algebra generated by elements

$$x_1, \ldots, x_n, x_1^*, \ldots, x_n^*$$

satisfying the relations (1) and (2) above.

In 1977 J. Cuntz introduced a class of C^*-algebras (now called Cuntz algebras) generated by n nonunitry isometries. Specifically, if $n > 1$ the C^*-algebra $\mathcal{O}_n$ is generated by isometries $s_1, \ldots, s_n$ satisfying

1. $s_i^* s_j = \delta_{ij} \, \mathrm{Id}$
2. $\displaystyle\sum_{i=1}^{n} s_i s_i^* = \mathrm{Id}.$

Note: $L_{\mathbb{C}}(1, n)$ is isomorphic to a dense $*$-subalgebra of $\mathcal{O}_n$.

The Cuntz algebras have been generalized in a number of ways, including (in the late 1990's) the graph C^*-algebras.

Definition

A *graph* (E^0, E^1, r, s) consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge.

A *path* in E is a sequence of edges $\alpha := e_1 \dots e_n$ with $r(e_i) = s(e_{i+1})$.

We write $r(\alpha) = r(e_n)$ and $s(\alpha) = s(e_1)$, and say α has length $|\alpha| = n$.

We consider vertices to be paths of length zero, with $s(v) = r(v) = v$.

The set of all paths is denoted E^*.

A *cycle* is a path α with $|\alpha| \geq 1$ and $s(\alpha) = r(\alpha)$.

Definition

If E is a graph, the *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by a *Cuntz-Krieger E-family*, which consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e : e \in E^1\}$ satisfying

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
2. $p_v = \sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^*$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$
3. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$.

If A is a C^*-algebra and $\{P_v, S_e\} \subseteq A$ are elements satisfying the above conditions, then there exists a unique $*$-homomorphism $\phi : C^*(E) \to A$ satisfying $\phi(p_v) = P_v$ and $\phi(s_e) = S_e$.

For a path $\alpha := e_1 \ldots e_n$ in E, we define $s_\alpha := s_{e_1} \ldots s_{e_n}$. We see that

$$C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}.$$

The graph E not only describes the relations that the generators of $C^*(E)$ satisfy, but also many C^*-algebraic properties of $C^*(E)$ correspond to graph properties of E.

C*(E)	**Properties of E**
Unital	finite number of vertices
Finite Dim.	finite graph with no cycles
Simple	(1) Every cycle has an exit (2) No saturated hereditary sets
Simple and Purely Infinite	(1) Every cycle has an exit (2) No saturated hereditary sets (3) Every vertex can reach a cycle
AF	no cycles

Recall that much of the structure of $C^*(E)$ is deduced from the Uniqueness Theorems.

Theorem (Gauge-Invariant Uniqueness)

Let E be a graph, and let $\gamma : \mathbb{T} \to \mathrm{Aut}\, C^*(E)$ be the gauge action on $C^*(E)$. Also let $\phi : C^*(E) \to A$ be a $*$-homomorphism between C^*-algebras. If $\beta : \mathbb{T} \to \mathrm{Aut}\, A$ is a gauge action on A and the following two conditions are satisfied

1. $\beta_z \circ \phi = \phi \circ \gamma_z$ *for all* $z \in \mathbb{T}$
2. $\phi(p_v) \neq 0$ *for all* $v \in E^0$

then ϕ is injective.

Theorem (Cuntz-Krieger Uniqueness)

Let E be a graph in which every cycle has an exit. If $\phi : C^*(E) \to A$ is a $*$-homomorphism between C^*-algebras with the property that $\phi(p_v) \neq 0$ for all $v \in E^0$, then ϕ is injective.

Inspired by the success of graph C^*-algebras, Gene Abrams and Gonzalo Aranda-Pino introduced Leavitt path algebras.

Definition

Given a graph $E = (E^0, E^1, r, s)$ and a field K, we let $(E^1)^*$ denote the set of formal symbols $\{e^* : e \in E^1\}$ (called *ghost edges*). The Leavitt path algebra $L_K(E)$ is the universal K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set $\{e, e^* : e \in E^1\}$ of elements satisfying

1. $s(e)e = er(e) = e$ for all $e \in E^1$
2. $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$
3. $e^*f = \delta_{e,f}\, r(e)$ for all $e, f \in E^1$
4. $v = \sum_{s(e)=v} ee^*$ when $0 < |s^{-1}(v)| < \infty$.

We see that

$$L_K(E) = \text{span}_K\{\alpha\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}.$$

When $K = \mathbb{C}$, we may define a conjugate-linear involution $*$ on $L_{\mathbb{C}}(E)$ by

$$\left(\sum_{i=1}^{n} \lambda_i \alpha_i \beta_i^*\right)^* = \sum_{i=1}^{n} \overline{\lambda_i} \beta_i \alpha_i^*,$$

which makes $L_{\mathbb{C}}(E)$ into a $*$-algebra.

$C^*(E)$	$L_K(E)$	Properties of E
Unital	Unital	finite number of vertices
Finite Dim.	Finite Dim.	finite graph with no cycles
Simple	Simple	(1) Every cycle has an exit (2) No saturated hereditary sets
Simple and Purely Infinite	Simple and Purely Infinite	(1) Every cycle has an exit (2) No saturated hereditary sets (3) Every vertex can reach a cycle
AF	ultramatricial	no cycles

Many theorems from each class seem to have a corresponding theorem in the other, and the graph theoretic properties on E equivalent to an algebraic property on $L_K(E)$ often seem to be the same as the those graph theoretic properties equivalent to the corresponding C^*-algebraic property of $C^*(E)$.

These similarities might suggest that such structural properties, once obtained on either the graph C^*-algebra side or on the Leavitt path algebra side, might then immediately be translated via some sort of Rosetta stone to the corresponding property on the other side.

Nonetheless, a vehicle to transfer information in this way remains elusive, and in fact, researchers seem uncertain how to even formulate conjectures that would lead to such a vehicle.

Structure theorems for Leavitt path algebras.

Theorem (Cuntz-Krieger Uniqueness for graph C^*-algebras)

Let E be a graph in which every cycle has an exit. If $\phi : C^*(E) \to A$ is a $*$-homomorphism between C^*-algebras with the property that $\phi(p_v) \neq 0$ for all $v \in E^0$, then ϕ is injective.

A version of this can be proven for Leavitt path algebras.

Theorem (Cuntz-Krieger Uniqueness for Leavitt path algebras)

Let E be a graph in which every cycle has an exit. If $\phi : L_K(E) \to R$ is a homomorphism between rings with the property that $\phi(v) \neq 0$ for all $v \in E^0$, then ϕ is injective.

What about the Gauge-Invariant Uniqueness Theorem?

Theorem (Gauge-Invariant Uniqueness)

Let E be a graph, and let $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ be the gauge action on $C^*(E)$. Also let $\phi : C^*(E) \to A$ be a $*$-homomorphism between C^*-algebras. If $\beta : \mathbb{T} \to \operatorname{Aut} A$ is a gauge action on A and the following two conditions are satisfied

1. $\beta_z \circ \phi = \phi \circ \gamma_z$ *for all* $z \in \mathbb{T}$
2. $\phi(p_v) \neq 0$ *for all* $v \in E^0$

then ϕ is injective.

How can a version of this theorem be obtained for Leavitt path algebras? There is no natural action of $\mathbb{T}$ on $L_K(E)$.

What about an action of K^* on $L_K(E)$? This does not work.

The correct thing to do is look at the graded structure of $L_K(E)$.

A ring R is $\mathbb{Z}$*-graded* if it contains a collection of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ with $R = \bigoplus R_n$ as left R-modules and $R_m R_n \subseteq R_{m+n}$.

An ideal $I \triangleleft R$ is $\mathbb{Z}$*-graded* if $I = \bigoplus_{n\in\mathbb{Z}} (I \cap R_n)$.

A ring homomorphism $\phi : R \to S$ is $\mathbb{Z}$*-graded* if $\phi(R_n) \subseteq S_n$ for all $n \in \mathbb{Z}$.

For $n \in \mathbb{Z}$, let

$$L_K(E)_n := \text{span}\{\alpha\beta^* : \alpha \text{ and } \beta \text{ are paths with } |\alpha| - |\beta| = n\}.$$

This gives a $\mathbb{Z}$-grading on $L_K(E)$.

We can prove the following "Graded Uniqueness Theorem".

Theorem (Graded Uniqueness Theorem)

Let E be a graph, and let $L_K(E)$ have the $\mathbb{Z}$-grading described above. Also let $\phi : L_K(E) \to R$ be a homomorphism between rings. If the following two conditions are satisfied

1. *R has a $\mathbb{Z}$-grading and $\phi(L_K(E)_n) \subseteq R_n$*
2. *$\phi(v) \neq 0$ for all $v \in E^0$*

then ϕ is injective.

This Graded Uniqueness Theorem allows us to prove that the graded ideals in $L_K(E)$ correspond to pairs (H, S) where H is a saturated hereditary subset of vertices and S is a subset of breaking vertices.

Furthermore, if E is row-finite (there are no vertices that emit an infinite number of edges) then there are no breaking vertices and the map

$$H \mapsto I_H := \langle \{v : v \in H\} \rangle$$

is a lattice isomorphism from saturated hereditary subsets of vertices in E onto graded ideals of $L_K(E)$.

Also, in this case, we have

$$L_K(E)/I_H \cong L_K(E \setminus H)$$

and

$$I_H \text{ is Morita equivalent to } L_K(E_H).$$

Remark: If E is a graph, and $\{p_v, s_e : v \in E^0, e \in E^1\}$ is a generating Cuntz-Krieger family for $C^*(E)$, then the p_v's and s_e's satisfy the relations for the generators of a Leavitt algebra, and we get a homomorphism (in fact, a $*$-homomorphism)

$$\iota_E : L_{\mathbb{C}}(E) \to C^*(E)$$

mapping $v \mapsto p_v$ and $e \mapsto s_e$.

It can be shown (using the Graded Uniqueness Theorem) that ι_E is injective. Thus if we write

$$C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\},$$

then $L_{\mathbb{C}}(E)$ may be identified with the dense $*$-subalgebra

$$\operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}.$$

The fact that $L_{\mathbb{C}}(E)$ is a dense $*$-subalgebra of $C^*(E)$ does not explain the similarities between $L_{\mathbb{C}}(E)$ and $C^*(E)$. It is possible for a C^*-algebra to have vastly different properties from a dense $*$-subalgebra.

Different graphs can give rise to the same Leavitt path algebra. Similarly, different graphs can give rise to the same C^*-algebra. Given that the graph completely determines the associated Leavitt path algebra, and the same properties of the graph that determine each algebraic properties tend to determine the corresponding C^*-algebraic property of the graph C^*-algebra, it is natural to ask the following:

Question: If E and F are graphs and $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$, then is it the case that $C^*(E) \cong C^*(F)$?

We need to be a bit more careful about what we mean by $\cong$.

Question: If E and F are graphs and $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ (as $*$-algebras), then is it the case that $C^*(E) \cong C^*(F)$ (as $*$-algebras)?

Suppose $\phi : L_{\mathbb{C}}(E) \to L_{\mathbb{C}}(F)$ is an algebra $*$-homomorphism.

Recall we have injective $*$-homomorphisms $\iota_E : L_{\mathbb{C}}(E) \to C^*(E)$ and $\iota_F : L_{\mathbb{C}}(F) \to C^*(F)$.

If

$$\{v, e, e^* : v \in E^0, e \in E^1\}$$

is a generating E-family for $L_{\mathbb{C}}(E)$, then

$$\{\phi(v), \phi(e), \phi(e^*) : v \in E^0, e \in E^1\}$$

is a Leavitt E-family in $L_{\mathbb{C}}(F)$. Also,

$$\{\iota_F(\phi(v)), \iota_F(\phi(e)) : v \in E^0, e \in E^1\}$$

is a Cuntz-Krieger E-family in $C^*(F)$.

Thus we obtain a $*$-homomorphism $\overline{\phi} : C^*(E) \to C^*(F)$ making

$$\begin{array}{ccc} C^*(E) & \xrightarrow{\overline{\phi}} & C^*(F) \\ \uparrow \iota_E & & \uparrow \iota_F \\ L_{\mathbb{C}}(E) & \xrightarrow{\phi} & L_{\mathbb{C}}(F) \end{array}$$

commute. Moroever, if ϕ is an algebra $*$-isomorphism, then so is $\overline{\phi}$.

Thus we have proven

Theorem

If E and F are graphs and $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ (as $*$-algebras), then $C^*(E) \cong C^*(F)$ (as $*$-algebras). Moreover, any $*$-isomorphism from $L_{\mathbb{C}}(E)$ to $L_{\mathbb{C}}(F)$ lifts to a $*$-isomorphism from $C^*(E)$ to $C^*(F)$.

A similar argument shows

Theorem

If E and F are graphs and $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ (as $*$-rings), then $C^*(E) \cong C^*(F)$ (as $*$-algebras).

So if the $*$-structure is preserved, things are okay. In fact, we can extend algebra $*$-homomorphisms from Leavitt path algebras to graph C^*-algebras.

What about when the $*$-structure is not preserved? We make two conjectures.

Conjecture 1: If E and F are graphs and $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ (as rings), then $C^*(E) \cong C^*(F)$ (as $*$-algebras).

Conjecture 2: If E and F are graphs and $L_{\mathbb{C}}(E)$ is Morita equivalent to $L_{\mathbb{C}}(F)$ (as rings), then $C^*(E)$ is Morita equivalent to $C^*(F)$ (as C^*-algebras).

We can verify the two conjectures in the case when the graphs have no cycles (equivalently, the associated Leavitt path algebras are ultramatricial; equivalently, the associated graph C^*-algebras are AF-algebras).

We can also verify the two conjectures in the case when the associated Leavitt path algebras are simple (equivalently, the associated graph C^*-algebras are simple).

We accomplish this with K-theory classification theorems.

The K-theory of a graph C^*-algebra can be computed.
If E is a graph with no sinks and no infinite emitters, let A_E be the square matrix indexed by vertices with

$$A_E(v, w) := \text{ number of edges from } v \text{ to } w \text{ }\}.$$

Consider the map

$$A_E^t - I : \bigoplus_{E^0} \mathbb{Z} \longrightarrow \bigoplus_{E^0} \mathbb{Z}.$$

Then

$$K_0(C^*(E)) \cong \operatorname{coker} A_E^t - I$$

$$K_1(C^*(E)) \cong \ker A_E^t - I.$$

Also,

$$K_0(C^*(E))^+ = \left\{ \left[\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \right]_0 : x_i \in \mathbb{N} \right\}$$

and

$$[1]_0 = \left[\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right]_0 .$$

Elliott's Theorem applies to dense $*$-subalgebras. Also the K_0-group of $L_K(E)$ is the same as the K_0-group of $C^*(E)$.

Theorem ((Elliott))

Let A and B be AF C^*-algebras, let R be a dense ultramatricial $*$-subalgebra of A, and let S be a dense ultramatricial $*$-subalgebra of B. Then the following are equivalent:

1. $A \cong B$ *(as $*$-algebras)*
2. $R \cong S$ *(as $*$-algebras)*
3. $R \cong S$ *(as algebras)*
4. $(K_0(A), K_0(A)^+, \Sigma(A)) \cong (K_0(B), K_0(B)^+, \Sigma(B))$
5. $(K_0(R), K_0(R)^+, \Sigma(R)) \cong (K_0(S), K_0(S)^+, \Sigma(S))$

Thus for graphs with no cycles, the complex Leavitt path algebras are classified by their K_0-groups.

Theorem

If E and F are graphs with no cycles, then $C^*(E) \cong C^*(F)$ (as $*$-algebras) if and only if $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ (as algebras).

This is one part of the simple case. For the general case, we cannot obtain an "if and only if" statement.

Let E and F be graphs such that $L_K(E)$ and $L_K(F)$ are either both ultramatricial or both simple.

Then . . .

$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ (as algebras)

$\Longrightarrow$ algebraic K-groups of $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are isomorphic

$\Longrightarrow$ topological K-groups of $C^*(E)$ and $C^*(F)$ are isomorphic (this takes some work)

$\Longrightarrow$ $C^*(E) \cong C^*(F)$ (as $*$-algebras)

Some results about Morita equivalence

Recall: Two unital rings are defined to be *Morita equivalent* if and only if their categories of left modules are equivalent if and only if there exists an equivalence bimodule between the rings.

The notion of Morita equivalence has also been extended by Abrams (1983) to rings with local units. Leavitt path algebras have local units.

Rieffel developed a notion of Morita equivalence for C^*-algebras:

Two C^*-algebras are *Morita equivalent* if and only if their categories of Hermitian modules are equivalent.

Two C^*-algebras are *strongly Morita equivalent* if there exists an equivalence C^*-bimodule between them.

Two stabilization theorems:

Theorem (Brown-Green-Rieffel)

Let A and B be C^*-algebras, each with a countable approximate identity. Then $A \sim_{SME} B$ if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.

Theorem (Stephenson, Abrams-T)

Suppose that R and S are rings, each with local units. Then $R \sim_{ME} S$ if and only if $M_\infty(R) \cong M_\infty(S)$.

Using these stabilization theorems, we can prove a Morita equivalence version of Elliott's Theorem.

Theorem

Let A and B be AF C^*-algebras, let R be a dense ultramatricial $*$-subalgebra of A, and let S be a dense ultramatricial $*$-subalgebra of B. Then the following are equivalent:

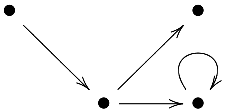
1. $A \sim_{SME} B$
2. $R \sim_{ME} S$
3. $(K_0(A), K_0(A)^+) \cong (K_0(B), K_0(B)^+)$
4. $(K_0(R), K_0(R)^+) \cong (K_0(S), K_0(S)^+)$

Given a graph E, let SE be the graph formed attaching an infinite "head" to each vertex v

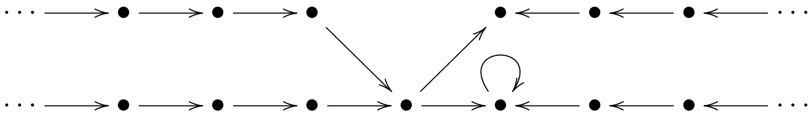
$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow v$$

to E. We call SE the *stabilization* of E.

Example: If E is the graph



then SE is the graph



Lemma: If E is a graph, then

(1) $L_{\mathbb{C}}(SE) \cong M_\infty(L_{\mathbb{C}}(E))$ (as $*$-algebras)

(2) $C^*(SE) \cong C^*(E) \otimes \mathcal{K}$ (as $*$-algebras)

Let E and F be graphs such that $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are either both AF or both simple.

Then . . .

$$L_{\mathbb{C}}(E) \sim_{ME} L_{\mathbb{C}}(F)$$

$$\Longrightarrow M_\infty(L_{\mathbb{C}}(E)) \cong M_\infty(L_{\mathbb{C}}(F)) \qquad \text{[stabilization result]}$$

$$\Longrightarrow L_{\mathbb{C}}(SE) \cong L_{\mathbb{C}}(SF) \text{ (as algebras)} \qquad \text{[by lemma]}$$

$$\Longrightarrow C^*(SE) \cong C^*(SF) \text{ (as $*$-algebras)}$$

$$\Longrightarrow C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K} \qquad \text{[by lemma]}$$

$$\Longrightarrow C^*(E) \sim_{SME} C^*(F) \qquad \text{[by Brown-Green-Reiffel].}$$

Some differences between Leavitt path algebras and graph C^*-algebras

- There is an example of a graph E for which the Leavitt path algebra $L_K(E)$ is prime (i.e., $\{0\}$ is a prime ideal), but $C^*(E)$ is not prime (i.e., $\{0\}$ is not a prime ideal).

- There is an example of a graph E for which the Leavitt path algebra $L_K(E)$ has stable rank 2 , but $C^*(E)$ has stable rank 1.

Carlsen and Ortega have described an algebraic analogue of the Cuntz-Pimsner algebras.

Given a ring R, an *R-system* is a triple (P, Q, ψ) where P and Q are R-bimodules and $\psi : P \otimes Q \to R$ is an R-bimodule homomorphism.

The algebraic Cuntz-Pimsner ring $\mathcal{O}_{(P,Q,\psi)}$ generalizes

- the Leavitt path algebra
- the crossed product of a ring by an automorphism
- the fractional skew monoid ring of a corner isomorphism

Moreover, there is a Graded Uniqueness Theorem for $\mathcal{O}_{(P,Q,\psi)}$ and one can classify the graded ideals of $\mathcal{O}_{(P,Q,\psi)}$ in terms of pairs of ideals in R.