Chaotic behavior in a forecast model

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March 19, 2016

Abstract

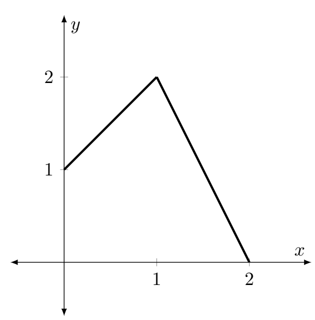
We examine a certain interval map, called the weather map, that has been used by previous authors as a toy model for weather forecasting. We prove that the weather map is topologically mixing and satisfies Devaney's definition of chaos.

1 Introduction

In the subject of dynamical systems there has been a great deal of progress developing abstract theories that describe the behavior of systems satisfying certain hypotheses, but relatively less development establishing that various systems satisfy these hypotheses. As a result, in recent years there has been a great deal of interest in examining particular examples of dynamical systems and determining their properties.

An example that has attracted recent attention is a "Toy Forecast Model" described in an article by Sadowski in the December 2012 issue of the *American Mathematical Monthly* [7]. This model involves a function $f : [0,2] \to [0,2]$ defined by

$$f(x) = \begin{cases} x+1 & \text{if } 0 \le x \le 1 \\ 4-2x & \text{if } 1 < x \le 2 \end{cases}$$



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‡Research supported by the Provost's Undergraduate Research Scholarship (PURS) at the University of Houston

and we shall refer to f as the *weather map*. The weather map was designed to give a simplified example describing how weather can change from one day to the next: a sunny day is labeled 0, a cloudy day is labeled 1, and a rainy day is labeled 2. A number $x \in [0,2]$ is considered as a position on this spectrum of sunny to rainy, and if x denotes today's weather, then the weather for the following days is given by the sequence $f(x), f^2(x), f^3(x), \ldots$, where we define $f^{k+1}(x) := f(f^k(x))$ for $k \geq 1$. Such a model is, of course, not realistic, but instead meant to provide a simplified deterministic model that can mimic sudden changes, similar to the changes that occur with daily weather. One can see that $x = 0$ is periodic, with $f(0) = 1$, $f(1) = 2$, and $f(2) = 0$. In his analysis in [7], Sadowski observed that when the initial weather is a dyadic rational; i.e., $x = a/2^n \in [0,2]$ for $a \in \mathbb{N}$, then the weather would eventually be equal to one of the values in $\{0,1,2\}$, and he colored the point x white, grey, or black, depending on whether the first integer value obtained by $f^k(x)$ was 0, 1, or 2, respectively. Sadowski showed that for every $n \in \mathbb{N}$ the numbers $0/2^n, 1/2^n, 2/2^n, \ldots, 2^{n+1}/2^n$ are colored in such a way that every three consecutive terms are of three different colors. This shows that the weather function exhibits a *sensitive dependence on initial conditions*: it is possible to have days of arbitrarily close weather that on some day in the future will produce days of drastically different weather.

Our objective in this article is to provide a proof that the weather map is topologically mixing and chaotic. While there is no universally accepted definition of chaos, one very popular definition is due to Devaney. A map $f : X \to X$ is chaotic in Devaney's definition if all of the following three properties hold: (1) f is topologically transitive, (2) the periodic points of f are dense in X, and (3) f has sensitive dependence on initial conditions. It is known that in many situations these conditions are not independent: If f is continuous and if X is a metric space with no isolated points, then we have $((1)+(2)) \implies (3)$, and if X is an interval of $\mathbb{R}$, then we have topologically mixing $\implies (1) \implies ((2)+(3))$. Thus in many situations, one only needs to establish a subset of the desired properties.

Our inverstigation of the weather map requires a careful consideration of cases: while the weather map is expanding on $[1,2]$, it is isometric on $[0,1]$, and hence one must do a careful analysis of how the weather maps causes the intervals $[0,1]$ and $[1,2]$ to interact. After we show that the weather map is topologically mixing, we apply well-known results to conclude that the weather map satisfies all three conditions of Devaney's definition of chaos.

2 Preliminary Definitions and Notation

We denote the real numbers by $\mathbb{R}$, the natural numbers by $\mathbb{N}$, and mention that $\mathbb{N} = \{1, 2, \ldots\}$; so, in particular, 0 is not a natural number. We use the term *interval* to mean a subset of the real numbers with the property that for any two real numbers in the set all numbers between these two numbers are also in the set. (This property is also sometimes called "convex".) Note that intervals of the real numbers need not be open, closed, or finite; e.g., $[1,2]$, $[3,7)$, $(0,1)$, and $[5,\infty)$ are all examples of intervals.

If X is a set and $f : X \to X$, then we define $f^0 : X \to X$ to be the identity function on X, and for any $k \in \mathbb{N}$ we define $f^k : X \to X$ recursively by $f^k := f \circ f^{k-1}$. Note that $f^1 = f$, and f^k is the k-fold composition of f. For any $k \in \mathbb{N}$ and any subset $S \subseteq X$, we also define $f^{-k}(S) = \{x \in X : f^k(x) \in S\}$.

Definition 1. *Let X be a topological space and let $f : X \to X$ be a function. We say that $x \in X$ is* periodic *if $f^k(x) = x$ for some $k \in \mathbb{N}$. In this case we say x has* period *k, and the smallest value of $k \in \mathbb{N}$ for which $f^k(x) = x$ is called the* least period *of x.*

Definition 2. *If X is a topological space and $f : X \to X$ is a function, we say f is* topologically transitive *if whenever U and V are nonempty open subsets of X, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$*

Definition 3. *If X is a topological space and $f : X \to X$ is a function, we say f is* topologically mixing *if whenever U and V are nonempty open subsets of X, there exists a $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.*

Remark 1. *It follows immediately from the definitions that topologically mixing implies topologically transitive. However, the converse does not hold: an irrational rotation of the circle is topologically transitive (the orbit of a small open interval will eventually intersect any other small open interval), but not topologically mixing (a rotating small interval will typically leave another interval for a while before returning).*

Definition 4. *If (X,d) is a metric space and $f : X \to X$, we say f is* sensitive to initial conditions *if there exists a $\delta > 0$ such that, for any $x \in X$ and any open set U containing x, there exists a $y \in U$ and an $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.*

Definition 5 (Devaney's Definition of Chaos [4])**.** *Let (X,d) be a metric space. We say a function $f : X \to X$ is* chaotic *(or* exhibits chaos*) if the following three conditions are satisfied:*

(1) f is topologically transitive,

(2) the periodic points of f are dense in X, and

(3) f has sensitive dependence on initial conditions.

In many situations these three conditions are not independent, and we examine the relationships among them in the next section.

3 Relationships among the conditions in Devaney's Definition

In this section we give a short description of the relationships among topologically mixing and the three properties of Devaney's definition, emphasizing various hypotheses under which one condition of Definition 5 will follow from others. We use the same numbering of properties as in Definition 5 for our notation.

It follows from the definitions that we always have the implication

$$\text{topologically mixing} \implies (1)$$

and the example of irrational rotation of the circle shows the converse implication does not hold. The following two propositions describe situations in which we have additional implications.

Banks, Brooks, Cairns, Davis, and Stacey proved the following proposition in [1].

Proposition 1. *Let (X, d) be a metric space with no isolated points, and let $f : X \to X$ be a continuous function. If f is topologically transitive and the set of periodic points of f is dense in X, then f has sensitive dependence on initial conditions.*

The next proposition was first proven by Block and Coppel [3, Lemma 41 of Chapter IV.5]. However, a highly simplified proof was given by Vellekoop and Berglund in [8], and Vellekoop and Berglund's proof was adapted slightly in [5, Theorem 3.6].

Proposition 2. *Let I be an interval of $\mathbb{R}$, and let $f : I \to I$ be continuous. If f is transitive, then the set of periodic points of f is dense in I.*

Proposition 1 and Proposition 2 describe two important situations in which we have implications: metric spaces with no isolated point and intervals of real numbers. We summarize the consequences of these propositions in the next two remarks.

Remark 2 (Metric Spaces with No Isolated Points)**.** *If $f : X \to X$ is continuous and X is a metric space with no isolated points, Proposition 1 shows*

$$((1) \textit{ and } (2)) \implies (3).$$

Remark 3 (Intervals of the Real Numbers)**.** *If $f : I \to I$ is continuous and I is an interval of $\mathbb{R}$, then Proposition 1 and Proposition 2 show*

$$\textit{topologically mixing} \implies (1) \implies ((2) \textit{ and } (3)).$$

Remark 4 (Counterexamples)**.** *The following four examples show the converse of the implications in Remark 2 and Remark 3 do not hold.*

- *Example 1: The identity function $id : I \to I$ is a continuous function on an interval that satisfies (2) but not (3)*
- *Example 2: [8, p.355] gives an example of a continuous function $f : I \to I$ on an interval that satisfies (3) but not (2).*

- *Example 3: [8, p.354] gives an example of a continuous function $f : I \to I$ on an interval that satisfies (2) and (3), but not (1).*
- *Example 4: It is shown in [6, Theorem 6.1.2] that if I is a closed bounded interval, then $f : I \to I$ is topologically mixing if and only if $f^2 : I \to I$ is topologically transitive. The function described in [2, Example 3] is a continuous function $f : I \to I$ on an closed bounded interval I such that f is topologically transitive, but f^2 is not topologically transitive. In particular, f is topologically transitive, but not topologically mixing.*

Example 2 and Example 3 show that the converse of implication in Remark 2 does not hold, and moreover, that even when f is a continuous map on an interval, (3) implies neither (2) nor (1). Example 3 shows that the converse of the second implication in Remark 3 does not hold, and moreover, Example 1 and Example 2 show that neither of (2) and (3) implies the other in this case. Finally, Example 4 shows that even when f is a continuous map on an interval, (1) does not imply topological mixing.

4 The Weather Map is Topologically Mixing and Chaotic

Our goal in this section is to show the weather map is topologically mixing and satisfies Devaney's definition of chaos.

Definition 6. *For an interval $I \subseteq \mathbb{R}$, let $m(I)$ denote the Lebesgue measure (i.e., length) of I. In particular, if I is a finite interval, then $m(I)$ is equal to the difference of the right endpoint minus the left endpoint of I. Hence*

$$m((a,b)) = m([a,b]) = m((a,b]) = m([a,b)) = b - a.$$

Lemma 1. *Let $f : [0,2] \to [0,2]$ be the weather map, and let $I \subseteq [0,2]$ be an interval. Then $f^n(I)$ is an interval for all $n \in \mathbb{N}$, and there exists $N \in \mathbb{N}$ such that $m(f^N(I)) \geq \min\{2m(I), 2\}$. Moreover, we can always choose $N \leq 4$.*

Proof: Since the weather map f is continuous, f^n is continuous for all $n \in \mathbb{N}$, and hence by the intermediate value theorem f^n takes connected sets to connected sets. Thus $f^n(I)$ is an interval for any interval $I \subseteq [0,2]$.

Let I be an interval, and suppose that a is the left endpoint of I and b is the right endpoint of I. Then $m(I) = b - a$. Consider three cases.

Case I: $b \leq 1$.

In this case $I \subseteq [0,1]$, and since $f(x) = x + 1$ for all $x \in I$, it follows that $f(I)$ is an interval with left endpoint $a + 1$ and right endpoint $b + 1$. Thus $f(I) \subseteq [1,2]$. Since $f(x) = 4 - 2x$ for all $x \in [1,2]$, $f^2(I) = f(f(I))$ is an interval with left endpoint $2 - 2b$ and right endpoint $2 - 2a$. Thus

$$m(f^2(I)) = (2-2a) - (2-2b) = 2b - 2a = 2(b-a) = 2m(I) \geq \min\{2m(I), 2\}$$

and the claim holds with $N := 2$.

CASE II: $1 \le a$.

In this case $I \subseteq [1,2]$, and since $f(x) = 4-2x$ for all $x \in [1,2]$, $f(I))$ is an interval with left endpoint $4-2b$ and right endpoint $4-2a$. Thus

$$m(f(I)) = (4-2a) - (4-2b) = 2b - 2a = 2(b-a) = 2m(I) \ge \min\{2m(I), 2\}$$

and the claim holds with $N := 2$.

CASE III: $a \le 1$ and $1 \le b$.

Since $m(I) = b-a = (b-1)+(1-a)$, either $b-1 \ge m(I)/2$ or $1-a \ge m(I)/2$. Consider the following three subcases.

SUBCASE III(i): $b-1 \ge m(I)/2$.

Since $b \ge 1$ and $f(x) = 4-2x$ for all $x \ge 1$, we see that $f([1,b)) = (4-2b, 2]$. If $4-2b < 1$, then $f(4-2b, 2] = [0,2]$. If $4-2b \ge 1$, then it follows from Case II that

$$m(f^2(4-2b, 2]) \ge \min\{2m((4-2b,2]), 2\} = \min\{4b-4, 2\}.$$

Thus in either situation, we have

$$\begin{aligned} m(f^2(4-2b,2]) \ge \min\{4b-4,2\} &= \min\{4(b-1),2\} \\ &\ge \min\{4(m(I)/2),2\} = \min\{2m(I),2\} \end{aligned}$$

Hence

$$m(f^3(I)) \le m(f^3([1,b)) = m(f^2((4-2b,2])) \ge \min\{2m(I),2\}$$

and the claim holds with $N := 3$.

SUBCASE III(ii): $1-a \ge m(I)/2$ and $2-2a > 1$.

Since $a \le 1$ and $f(x) = x+1$ for all $x \le 1$, we have $f((a,1]) = (a+1, 2]$ with $a+1 \ge 1$. Since $f(x) = 4-2x$ for all $x \ge 1$, we have $f^2((a,1]) = [0, 2-2a)$. Because $2-2a \ge 1$ in this case, it follows that $[0,1] \subseteq f^2((a,1])$ and hence

$$[0,2] = f([1,2]) = f^2([0,1]) \subseteq f^2(f^2(a,1])) = f^4((a,1]) \subseteq f^4(I)$$

so that

$$m(f^4(I)) \ge m([0,2]) = 2 \ge \min\{2m(I),2\}$$

and the claim holds with $N := 4$.

SUBCASE III(iii): $1-a \ge m(I)/2$ and $2-2a < 1$.

Since $a \le 1$ and $f(x) = x+1$ for all $x \le 1$, we have $f((a,1]) = (a+1,2]$ with $a+1 \ge 1$. Since $f(x) = 4-2x$ for all $x \ge 1$, we have $f^2((a,1]) = [0,2-2a)$. Because $2-2a \le 1$ in this case, it follows from Case I that $m(f^2([0,2-2a))) \ge \min\{2m([0,2-2a)),2\} = \min\{4-4a,2\}$. Hence

$$\begin{aligned} m(f^4(I)) \ge m(f^4((a,1])) &= m(f^2(f^2((a,1]))) = m(f^2([0,2-2a))) \\ &\ge \min\{4-4a,2\} = 4-4a = 4(1-a) \ge 4(m(I)/2) \\ &= 2m(I) \ge \min\{2m(I),2\} \end{aligned}$$

and the claim holds with $N := 4$. QED

Lemma 2. *Let $f : [0,2] \to [0,2]$ be the weather map, and let $I \subseteq [0,2]$ be an interval. Then $f^n(I)$ is an interval for all $n \in \mathbb{N}$, and for every $M \in N$ there exists $N \in \mathbb{N}$ such that $m(f^N(I)) \geq \min\{2^M m(I), 2\}$. Moreover, we can always choose $N \leq 4M$.*

Proof: We proceed by induction on M. If $M = 1$, the result follows from Lemma 1. For the inductive step, let $M \in \mathbb{N}$ and suppose the claim holds for $M - 1$. Then there exists $N' \in \mathbb{N}$ with $m(f^{N'}(I)) \geq \min\{2^{M-1}m(I), 2\}$ and $N' \leq 4(M-1)$. Applying Lemma 1 to the interval $f^{N'}(I)$, there exists $N'' \in \mathbb{N}$ with $m(f^{N''}(f^{N'}(I))) \geq \min\{2 \cdot 2^{M-1}m(I), 2\}$ and $N'' \leq 4$. If we let $N := N' + N''$, then $m(f^N(I)) \geq \min\{2 \cdot 2^{M-1}m(I), 2\} = \min\{2^M m(I), 2\}$ and $N = N' + N'' \leq 4(M-1) + 4 = 4M$. QED

Lemma 3. *If U is a nonempty open subset of $[0,2]$, and $m := m(U)$, then there exists $N \in \mathbb{N}$ such that $(0,2) \subseteq f^N(U)$. Moreover, if U contains an interval I with $m(I) > 0$, then we may choose $N \leq 4\log_2(2/m(I)) + 4$.*

Proof: Since U is a nonempty open subset, then exists a nonempty open interval $I = (a,b)$ with $I \subseteq U$. Since $m(I) > 0$, we may choose a natural number M such that $\log_2(2/m(I)) \leq M < \log_2(2/m(I)) + 1$. By Lemma 2 there exists $N \leq 4M$ such that $f^N(I)$ is an interval and $m(f^N(I)) \geq \min\{2^M m(I), 2\}$. However,

$$2^M m(I) \geq 2^{\log_2(2/m(I))} m(I) = (2/m(I))m(I) = 2$$

so $m(f^N(I)) = 2$. Since $f^N(I)$ is an interval contained in $[0,2]$ with length 2, it follows that $(0,2) \subseteq f^N(I)$. Since $I \subseteq U$, we have $(0,2) \subseteq f^N(I) \subseteq f^N(U)$. Moreover, $N \leq 4M < 4(\log_2(2/m(I)) + 1) = 4\log_2(2/m(I)) + 4$. QED

Theorem 5. *If $f : [0,2] \to [0,2]$ is the weather map, then f is topologically mixing and f satisfies the three properties in Devaney's definition of chaos (see Definition 5).*

Proof: If U is a nonempty open set, it follows from Lemma 3 that there exists $N \in \mathbb{N}$ such that $(0,2) \subseteq f^N(U)$. Thus $(0,2) \subseteq f^n(U)$ for all $n \geq N$, and $f^n(U)$ intersects every nonempty open subset of $[0,2]$ nontrivially. Thus f is topologically mixing. It follows that f is also topologically transitive, and Proposition 1 and Proposition 2 imply that f satisfies the three properties in Devaney's definition of chaos (cf. Remark 3). QED

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