

Chaotic behavior in a forecast model

Michael Boyle ^{*} Mark Tomforde ^{†‡}

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Abstract

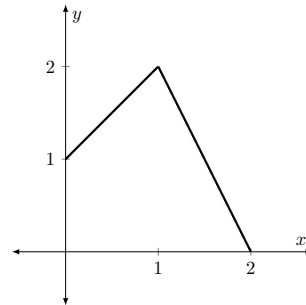
We examine a certain interval map, called the weather map, that has been used by previous authors as a toy model for weather forecasting. We prove that the weather map is topologically mixing and satisfies Devaney’s definition of chaos.

1 Introduction

In the subject of dynamical systems there has been a great deal of progress developing abstract theories that describe the behavior of systems satisfying certain hypotheses, but relatively less development establishing that various systems satisfy these hypotheses. As a result, in recent years there has been a great deal of interest in examining particular examples of dynamical systems and determining their properties.

An example that has attracted recent attention is a “Toy Forecast Model” described in an article by Sadowski in the December 2012 issue of the *American Mathematical Monthly* [7]. This model involves a function $f : [0, 2] \rightarrow [0, 2]$ defined by

$$f(x) = \begin{cases} x + 1 & \text{if } 0 \leq x \leq 1 \\ 4 - 2x & \text{if } 1 < x \leq 2 \end{cases}$$



^{*}Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA; email: michael.w.boyle0@gmail.com

[†]Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA; email: tomforde@math.uh.edu

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and we shall refer to f as the *weather map*. The weather map was designed to give a simplified example describing how weather can change from one day to the next: a sunny day is labeled 0, a cloudy day is labeled 1, and a rainy day is labeled 2. A number $x \in [0, 2]$ is considered as a position on this spectrum of sunny to rainy, and if x denotes today's weather, then the weather for the following days is given by the sequence $f(x), f^2(x), f^3(x), \dots$, where we define $f^{k+1}(x) := f(f^k(x))$ for $k \geq 1$. Such a model is, of course, not realistic, but instead meant to provide a simplified deterministic model that can mimic sudden changes, similar to the changes that occur with daily weather. One can see that $x = 0$ is periodic, with $f(0) = 1$, $f(1) = 2$, and $f(2) = 0$. In his analysis in [7], Sadowski observed that when the initial weather is a dyadic rational; i.e., $x = a/2^n \in [0, 2]$ for $a \in \mathbb{N}$, then the weather would eventually be equal to one of the values in $\{0, 1, 2\}$, and he colored the point x white, grey, or black, depending on whether the first integer value obtained by $f^k(x)$ was 0, 1, or 2, respectively. Sadowski showed that for every $n \in \mathbb{N}$ the numbers $0/2^n, 1/2^n, 2/2^n, \dots, 2^{n+1}/2^n$ are colored in such a way that every three consecutive terms are of three different colors. This shows that the weather function exhibits a *sensitive dependence on initial conditions*: it is possible to have days of arbitrarily close weather that on some day in the future will produce days of drastically different weather.

Our objective in this article is to provide a proof that the weather map is topologically mixing and chaotic. While there is no universally accepted definition of chaos, one very popular definition is due to Devaney. A map $f : X \rightarrow X$ is chaotic in Devaney's definition if all of the following three properties hold: (1) f is topologically transitive, (2) the periodic points of f are dense in X , and (3) f has sensitive dependence on initial conditions. It is known that in many situations these conditions are not independent: If f is continuous and if X is a metric space with no isolated points, then we have $((1) + (2)) \implies (3)$, and if X is an interval of \mathbb{R} , then we have topologically mixing $\implies (1) \implies ((2) + (3))$. Thus in many situations, one only needs to establish a subset of the desired properties.

In this paper we prove that the weather map is topologically mixing. Since it is defined on an interval of \mathbb{R} , this implies the weather map also satisfies Devaney's definition of chaos. However, rather than appeal to general theory to obtain these properties, we give a completely self-contained treatment of all the results needed. This allows readers of this paper to deduce the properties asserted from first principles and to not only prove — but also understand why — the weather map exhibits all three properties of Devaney's definition. After some preliminaries in Section 2, we describe the relationships among the three conditions in Devaney's definition of chaos in Section 3. In particular, we give self-contained proofs of the fact that $(1) + (2) \implies (3)$ when X is a metric space with no isolated points, and that $(1) \implies (2)$ when X is an interval of \mathbb{R} . These results are well known and we prove them by methods similar to the existing literature; however, we write the proofs to be completely self-contained and to have greater detail than what is in the literature; in particular, we write them at a level that is accessible to the average undergraduate reader.

In Section 4 we prove that the weather map is topologically mixing. This result requires a careful consideration of cases: while the weather map is expanding on $[1, 2]$, it is isometric on $[0, 1]$, and hence one must do a careful analysis of how the weather maps causes the intervals $[0, 1]$ and $[1, 2]$ to interact. Once we obtain that the weather map is topologically mixing, we apply our results from Section 3 to conclude that the weather map satisfies all three conditions of Devaney's definition of chaos.

2 Preliminary Definitions and Notation

We denote the real numbers by \mathbb{R} , the natural numbers by \mathbb{N} , and mention that $\mathbb{N} = \{1, 2, \dots\}$; so, in particular, 0 is not a natural number. If (X, d) is a metric space with metric d , then for any point $p \in X$ we let $B_r(p) := \{x \in X : d(p, x) < r\}$ denote the ball of radius r centered at p .

An *interval* is a subset of \mathbb{R} of one of the following eight types:

$$\begin{aligned} [a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} & [a, \infty) &:= \{x \in \mathbb{R} : a \leq x\} \\ (a, b) &:= \{x \in \mathbb{R} : a < x < b\} & (a, \infty) &:= \{x \in \mathbb{R} : a < x\} \\ [a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} & (\infty, b] &:= \{x \in \mathbb{R} : x \leq b\} \\ (a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} & (\infty, b) &:= \{x \in \mathbb{R} : x < b\} \end{aligned}$$

where $a, b \in \mathbb{R}$. The intervals of the form $[a, b]$, (a, b) , $[a, b)$, and $(a, b]$ are called *finite intervals*, the intervals of the form $[a, b]$, $[a, \infty)$, and $(\infty, b]$ are called *closed intervals*, and the intervals of the form (a, b) , (a, ∞) , and (∞, b) are called *open intervals*.

Notice that intersections of intervals are intervals, and that a subset of \mathbb{R} is an interval if and only if it has the property that for any two real numbers in the set all numbers between these two numbers are also in the set.

If X is a set and $f : X \rightarrow X$, then we define $f^0 : X \rightarrow X$ to be the identity function on X , and for any $k \in \mathbb{N}$ we define $f^k : X \rightarrow X$ recursively by $f^k := f \circ f^{k-1}$. Note that $f^1 = f$, and f^k is the k -fold composition of f . For any $k \in \mathbb{N}$ and any subset $S \subseteq X$, we also define $f^{-k}(S) = \{x \in X : f^k(x) \in S\}$.

Definition 1. Let X be a topological space and let $f : X \rightarrow X$ be a function. We say that $x \in X$ is *periodic* if $f^k(x) = x$ for some $k \in \mathbb{N}$. In this case we say x has *period* k , and the smallest value of $k \in \mathbb{N}$ for which $f^k(x) = x$ is called the *least period* of x . If $x \in X$, the *orbit* of x under f is the set

$$O(x) := \{x, f(x), f^2(x), f^3(x), \dots\}.$$

One can see that x is a periodic point, then $O(x)$ is a finite set of cardinality equal to the least period of x , and each element of $O(x)$ is also a periodic point. It is also straightforward to show that if x and y are periodic points, then either $O(x) \cap O(y) = \emptyset$ or $O(x) = O(y)$.

Definition 2. If X is a topological space and $f : X \rightarrow X$ is a function, we say f is *topologically transitive* if whenever U and V are nonempty open subsets of X , there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$

Definition 3. If X is a topological space and $f : X \rightarrow X$ is a function, we say f is topologically mixing if whenever U and V are nonempty open subsets of X , there exists a $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Remark 1. It follows immediately from the definitions that topologically mixing implies topologically transitive. However, the converse does not hold: an irrational rotation of the circle is topologically transitive (the orbit of a small open interval will eventually intersect any other small open interval), but not topologically mixing (a rotating small interval will typically leave another interval for a while before returning).

Definition 4. If (X, d) is a metric space and $f : X \rightarrow X$, we say f is sensitive to initial conditions if there exists a $\delta > 0$ such that, for any $x \in X$ and any open set U containing x , there exists a $y \in U$ and an $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

Definition 5 (Devaney's Definition of Chaos [4]). Let (X, d) be a metric space. We say a function $f : X \rightarrow X$ is chaotic (or exhibits chaos) if the following three conditions are satisfied:

- (1) f is topologically transitive,
- (2) the periodic points of f are dense in X , and
- (3) f has sensitive dependence on initial conditions.

In many situations these three conditions are not independent, and we examine the relationships among them in the next section.

3 Relationships among the conditions in Devaney's Definition

In this section we describe various hypotheses under which one condition of Definition 5 will follow from others.

3.1 Metric spaces without isolated points

We first prove that for a function on a metric space without isolated points, topologically transitive and dense periodic points implies sensitive dependence on initial conditions.

This result was first proven by Banks, Brooks, Cairns, Davis, and Stacey in [1]. Our proofs of Lemma 1, Lemma 2, and Proposition 1 are very similar to what is done in [1], but we strive to be more detailed in our presentation.

Lemma 1. Let (X, d) be a metric space with no isolated points, and let $f : X \rightarrow X$ be a continuous function. If f is topologically transitive and the set of periodic points of f is dense in X , then there exist periodic points $p_0, q_0 \in X$ such that $O(p_0) \cap O(q_0) = \emptyset$.

PROOF: Since the set of periodic points is dense in X , there exists a periodic point $p_0 \in X$. The fact p_0 is periodic implies $O(p_0)$ is a finite set. Let $r := \min\{d(p_0, p) : p \in O(p_0)\}$. Since X has no isolated points, and the periodic points are dense in X , there exists a periodic point $q_0 \in B_r(p_0)$ with $q_0 \neq p_0$. Since $B_r(p_0) \cap O(p_0) = \{p_0\}$ we conclude that $q_0 \notin O(p_0)$. Thus $q_0 \in O(q_0) \setminus O(p_0)$, and $O(q_0) \neq O(p_0)$. Since p_0 and q_0 are both periodic points, it follows that $O(p_0) \cap O(q_0) = \emptyset$. QED

Lemma 2. *Let (X, d) be a metric space, and let $f : X \rightarrow X$ be a continuous function with the property that there exist periodic points $p_0, q_0 \in X$ for f such that $O(p_0) \cap O(q_0) = \emptyset$. Then there exists $\epsilon_0 > 0$ such that for every $x \in X$ there is a periodic point $p \in X$ with $d(x, f^n(p)) \geq \epsilon_0$ for all $n \in \mathbb{N}$*

PROOF: Choose periodic points p_0 and q_0 such that $O(p_0) \cap O(q_0) = \emptyset$. Since p_0 and q_0 are periodic points, the orbits $O(p_0)$ and $O(q_0)$ are each finite sets. Thus the set $\{d(x, y) : x \in O(p_0) \text{ and } y \in O(q_0)\}$ is finite, and we may define

$$\epsilon_0 := \frac{1}{2} \min\{d(x, y) : x \in O(p_0) \text{ and } y \in O(q_0)\}.$$

Let $x \in X$. If $d(x, y) \geq \epsilon_0$ for all $y \in O(q_0)$, the claim holds for x by letting $p := q_0$. Otherwise, there exists $y \in O(q_0)$ such that $d(x, y) < \epsilon_0$, and for all $n \in \mathbb{N}$ we have $d(f^n(p_0), y) \leq d(f^n(p_0), x) + d(x, y)$. Thus $d(f^n(p_0), x) \geq d(f^n(p_0), y) - d(x, y) \geq 2\epsilon_0 - \epsilon_0 = \epsilon_0$ and the claim holds for x by letting $p := p_0$. QED

Proposition 1. *Let (X, d) be a metric space with no isolated points, and let $f : X \rightarrow X$ be a continuous function. If f is topologically transitive and the set of periodic points of f is dense in X , then f has sensitive dependence on initial conditions.*

PROOF: Lemma 1 and Lemma 2 imply that there exists $\epsilon_0 > 0$ such that for every $x \in X$ there is a periodic point $p \in X$ such that $d(x, f^n(p)) \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Let $\delta := \epsilon_0/4$. We shall show that f is sensitive to initial conditions using the value δ . Let $x \in X$ and let U be an open subset of X with $x \in U$. Choose $\epsilon > 0$ small enough that $\epsilon < \delta$ and $B_\epsilon(x) \subseteq U$. Since the periodic points of f are dense, there exists a periodic point $y \in B_\epsilon(x)$. Let us suppose y has period $k \in \mathbb{N}$. By the choice of ϵ_0 there exists a periodic point p with $d(x, f^n(p)) \geq \epsilon_0$ for all $n \in \mathbb{N}$. Let

$$V := \{z \in X : d(f^i(z), f^i(p)) < \epsilon \text{ for all } 1 \leq i \leq k\}.$$

It is straightforward to verify that $V = \bigcap_{i=1}^k f^{-i}(B_\epsilon(f^i(p)))$, and thus the continuity of f implies V is the finite intersection of open sets, and hence V is an open set. Since $B_\epsilon(x)$ and V are open sets, the fact f is topologically transitive implies that there exists $m \in \mathbb{N}$ such that $f^m(B_\epsilon(x)) \cap V \neq \emptyset$. Hence there exists $z \in B_\epsilon(x)$ such that $f^m(z) \in V$.

Choose $r \in \mathbb{N}$ such that $m/k < r \leq m/k + 1$. Then $m < kr \leq m + k$ and $0 < kr - m \leq k$. In particular, $kr - m \in \mathbb{N}$ and $kr - m \leq k$. Thus we have

$$\begin{aligned}
4\epsilon &= \epsilon_0 \\
&\leq d(x, f^{kr-m}(p)) && \text{(by choice of } p) \\
&\leq d(x, y) + d(y, f^{kr}(z)) + d(f^{kr}(z), f^{kr-m}(p)) && \text{(by triangle inequality)} \\
&\leq d(x, y) + d(y, f^{kr}(z)) + d(f^{kr-m}(f^m(z)), f^{kr-m}(p)) \\
&\leq d(x, y) + d(y, f^{kr}(z)) + \epsilon && \text{(since } f^m(z) \in V) \\
&\leq d(x, y) + d(f^{kr}(y), f^{kr}(z)) + \epsilon && \text{(since } y \text{ has period } k) \\
&\leq \epsilon + d(f^{kr}(y), f^{kr}(z)) + \epsilon && \text{(since } y \in B_\epsilon(x))
\end{aligned}$$

and rearranging terms gives $2\epsilon \leq d(f^{kr}(y), f^{kr}(z))$. Using the triangle inequality, we deduce

$$2\epsilon < d(f^{kr}(y), f^{kr}(z)) \leq d(f^{kr}(y), x) + d(x, f^{kr}(z)).$$

It follows that either $d(f^{kr}(y), x) \geq \epsilon$ or $d(x, f^{kr}(z)) \geq \epsilon$. Since $y \in B_\epsilon(x) \subseteq U$ and $z \in B_\epsilon(x) \subseteq U$, we have established f has sensitive dependence on initial conditions. QED

Remark 2. *Suppose (X, d) is a metric space and $f : X \rightarrow X$ is a continuous function that is topologically transitive and whose periodic points are dense. If $x \in X$ is an isolated point of X , then $\{x\}$ is an open set, and the denseness of periodic points implies that x is periodic. Moreover, in this case the fact f is topologically transitive implies that $X = O(x)$ is a finite set. Hence the hypothesis that X has no isolated points in Proposition 1 is equivalent to requiring that X is not a finite set.*

Remark 3. *An interesting aspect of Proposition 1 is that while topologically transitive and having dense periodic points are both topological properties, they are in this situation able to imply sensitive dependence on initial conditions, which is a metric property. Thus whether or not f satisfies Devaney's definition of chaos depends only on the topology of X and not the metric.*

3.2 Intervals of the real numbers

Next we prove that for a function on a (not necessarily closed and not necessarily finite) interval of the real numbers, f topologically transitive implies that the periodic points of f are dense in X . This result was first proven by Block and Coppel [3, Lemma 41 of Chapter IV.5]. However, a highly simplified proof was given by Vellekoop and Berglund in [8], and Vellekoop and Berglund's proof was adapted slightly in [5, Theorem 3.6]. In both [8] and [5] the result relies on a lemma (our Lemma 4), however in both cases the lemma was proven in a way that left out several details; there are multiple times in each proof where the authors claim that various results may be obtained by repeating an

argument similar to the one in their previous paragraphs with some obvious modifications. Rather than make similar statements, we have found that we can create an additional lemma (our Lemma 3) that may be applied repeatedly in the proof of Lemma 3 to avoid these vague justifications. We also found that by creating Lemma 5 we were able to give a slightly modified proof of the result in Proposition 2 that makes the argument more clear. We believe that our changes provide an easier to read and more transparent proof of the result in Proposition 2.

Lemma 3. *Let I be an interval of \mathbb{R} . Let $f : I \rightarrow I$ be continuous, and let $J \subseteq I$ be an interval that contains no periodic points of f . If $z \in J$ and $f(z) \in J$, then either $z < f(z) \leq f^k(z)$ for all $k \in \mathbb{N}$ or $f^k(z) \leq f(z) < z$ for all $k \in \mathbb{N}$.*

PROOF: Since J has no periodic points and $z \in J$, it follows that $f(z) \neq z$. Hence either $z < f(z)$ or $f(z) < z$. Let us first consider the case that $z < f(z)$. We shall prove that $z < f(z) \leq f^k(z)$ for all $k \in \mathbb{N}$ by induction on k . For the base case $k = 1$, we have $f^k(z) = f(z)$ and the claim holds trivially. For the inductive step suppose that $z < f(z) \leq f^k(z)$ for some $k \in \mathbb{N}$. Define $g : I \rightarrow \mathbb{R}$ by $g(x) := f^k(x) - x$. Then $g(z) = f^k(z) - z > 0$ by the inductive hypothesis. If it is the case that $g(f(z)) \leq 0$, then the intermediate value theorem implies that there exists $c \in [z, f(z)]$ such that $g(c) = 0$, and hence $f^k(c) = c$, contradicting the fact that $[z, f(z)] \subseteq J$ and there are no periodic points of f between z and $f(z)$. Hence we must have $g(f(z)) > 0$, and thus $f^{k+1}(z) > f(z)$. Therefor the claim holds for $k + 1$, and by the principle of mathematical induction, the claim holds for all $k \in \mathbb{N}$.

In the case that $f(z) < z$, a nearly identical induction argument can be used to show that $z > f(z) \geq f^k(z)$ for all $k \in \mathbb{N}$. QED

Lemma 4. *Let I be an interval of \mathbb{R} . Let $f : I \rightarrow I$ be continuous, and let $J \subseteq I$ be an interval that contains no periodic points of f . If $m, n \in \mathbb{N}$ with $m < n$, and $z \in J$ is a point with $f^m(z) \in J$ and $f^n(z) \in J$, then either $z < f^m(z) < f^n(z)$ or $z > f^m(z) > f^n(z)$.*

PROOF: Since J contains no periodic points, it follows that $f^m(z) \neq z$. Hence either $z < f^m(z)$ or $f^m(z) < z$. Let us first consider the case that $z < f^m(z)$. Define $g : I \rightarrow \mathbb{R}$ by $g(x) := f^m(x)$. Then $z \in J$, $g(z) \in J$, and $z < g(z)$, so it follows from Lemma 3 that $z < g(z) \leq g^k(z)$ for all $k \in \mathbb{N}$. Hence $z < f^{mk}(z)$ for all $k \in \mathbb{N}$. If we choose $k := n - m$, then $z < f^{m(n-m)}(z)$. We shall define $h : I \rightarrow \mathbb{R}$ by $h(x) = f^{n-m}(x)$, and observe that the previous sentence implies

$$z \leq h^m(z) \tag{1}$$

For the sake of contradiction, let us suppose that $f^n(z) \leq f^m(z)$. Then $h(f^m(z)) = f^{(n-m)}(f^m(z)) = f^n(z) \leq f^m(z)$, and since $f^m(z) \in J$ and $h(f^m(z)) = f^n(z) \in J$, it follows from Lemma 3 that $h^k(f^m(z)) \leq h(f^m(z)) < f^m(z)$ for all $k \in \mathbb{N}$. If we choose $k := m$, we obtain

$$h^m(f^m(z)) \leq f^m(z). \tag{2}$$

Define $p : I \rightarrow \mathbb{R}$ by $p(y) = h^m(y) - y$. Then (1) and (2) show that $p(y)$ is nonnegative at $y = z$ and $p(y)$ is nonpositive at $y = f^m(z)$. Thus by the intermediate value theorem, there exists $c \in [z, f^m(z)]$ such that $p(c) = 0$. But then $h^m(c) = c$, and $f^{m(n-m)}(c) = h^m(c) = c$, which contradicts the fact that $[z, f^m(z)] \subseteq J$ and there are no periodic points of f between z and $f^m(z)$. Therefore, we must have $f^m(z) < f^n(z)$ to avoid this contradiction, and we have shown that $z < f^m(z) < f^n(z)$.

In the case $f^m(z) < z$, a similar argument as above shows that $z > f^m(z) > f^n(z)$.

QED

Lemma 5. *Let X be a topological space, and let $f : X \rightarrow X$ be a continuous function that is topologically transitive. If U and V are nonempty open subsets of X , then for every $N \in \mathbb{N}$ there exists $n > N$ such that $f^n(U) \cap V \neq \emptyset$.*

PROOF: It suffices to prove that if $m \in \mathbb{N}$ and $f^m(U) \cap V \neq \emptyset$, then there exists $n > m$ such that $f^n(U) \cap V \neq \emptyset$.

If $f^m(U) \cap V \neq \emptyset$, then since $U \subseteq f^{-m}(V)$ and f^m is continuous, we may conclude that $f^{-m}(V)$ is a nonempty open set. Since f is topologically transitive, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap f^{-m}(V) \neq \emptyset$. It follows that

$$\emptyset \neq f^m(f^k(U) \cap f^{-m}(V)) \subseteq f^m(f^k(U)) \cap f^m(f^{-m}(V)) \subseteq f^{m+k}(U) \cap V$$

and hence if we let $n := m + k$ we have $n > m$ and $f^n(U) \cap V \neq \emptyset$. QED

Proposition 2. *Let I be an interval of \mathbb{R} , and let $f : I \rightarrow I$ be continuous. If f is transitive, then the set of periodic points of f is dense in I .*

PROOF: Suppose the set of periodic points of f is not dense in I . Then there exists a nonempty open interval $J \subseteq I$ such that J contains no periodic points of f . Since J is an open interval, there exist at least two distinct points in J , and hence we may choose nonempty open intervals $U \subseteq J$ and $V \subseteq J$ with $U \cap V = \emptyset$. Since f is topologically transitive, there exists $m \in \mathbb{N}$ such that $f^m(U) \cap V \neq \emptyset$. Choose $y \in U$ with $f^m(y) \in V$. Since f^m is continuous, there exists an open interval W with $y \in W \subseteq U$ and $f^m(W) \subseteq V$. In addition, since f is topologically transitive and W is a nonempty open set, Lemma 5 implies there exists $n > m$ such that $f^n(W) \cap W \neq \emptyset$. Choose $x \in W$ such that $f^n(x) \in W$. Then $0 < m < n$, and both $f^n(x) \in W$ and $x \in W$. However, $f^m(x) \in f^m(W) \subseteq V$, and hence $f^m(x) \notin W$. Since W is an interval, this implies $f^m(x)$ is not between x and $f^n(x)$. Since x , $f^m(x)$, and $f^n(x)$ are all elements of J , this contradicts Lemma 4. Thus the set of periodic points of f is dense in I . QED

3.3 Summary of the relationships

Here we give a short description of the consequences of our results in this section. If X is a topological space and $f : X \rightarrow X$ is continuous, we describe the relationships among topologically mixing and the three properties of Devaney's

definition. We use the same numbering of properties as in Definition 5 for our notation.

It follows from the definitions that we always have the implication

$$\text{topologically mixing} \implies (1)$$

and the example of irrational rotation of the circle shows the converse implication does not hold. The following two remarks contain situations in which we have additional implications.

Remark 4 (Metric Spaces with No Isolated Points). *If $f : X \rightarrow X$ is continuous and X is a metric space with no isolated points, Proposition 1 shows*

$$((1) \text{ and } (2)) \implies (3).$$

Remark 5 (Intervals). *If $f : I \rightarrow I$ is continuous and I is an interval of \mathbb{R} , then Proposition 2 and Proposition 1 show*

$$\text{topologically mixing} \implies (1) \implies ((2) \text{ and } (3)).$$

Remark 6 (Counterexamples). *The following four examples show the converse of the implications in Remark 4 and Remark 5 do not hold.*

- *Example 1: The identity function $\text{id} : I \rightarrow I$ is a continuous function on an interval that satisfies (2) but not (3)*
- *Example 2: [8, p.355] gives an example of a continuous function $f : I \rightarrow I$ on an interval that satisfies (3) but not (2).*
- *Example 3: [8, p.354] gives an example of a continuous function $f : I \rightarrow I$ on an interval that satisfies (2) and (3), but not (1).*
- *Example 4: It is shown in [6, Theorem 6.1.2] that if I is a closed bounded interval, then $f : I \rightarrow I$ is topologically mixing if and only if $f^2 : I \rightarrow I$ is topologically transitive. The function described in [2, Example 3] is a continuous function $f : I \rightarrow I$ on a closed bounded interval I such that f is topologically transitive, but f^2 is not topologically transitive. In particular, f is topologically transitive, but not topologically mixing.*

Example 2 and Example 3 show that the converse of implication in Remark 4 does not hold, and moreover, that even when f is a continuous map on an interval, (3) implies neither (2) nor (1). Example 3 shows that the converse of the second implication in Remark 5 does not hold, and moreover, Example 1 and Example 2 show that neither of (2) or (3) implies the other in this case. Finally, Example 4 shows that even when f is a continuous map on an interval, (1) does not imply topological mixing.

4 The Weather Map is Topologically Mixing and Chaotic

Our goal in this section is to show the weather map is topologically mixing and satisfies Devaney's definition of chaos.

Definition 6. For an interval $I \subseteq \mathbb{R}$, let $m(I)$ denote the Lebesgue measure (i.e., length) of I . In particular, if I is a finite interval, then $m(I)$ is equal to the difference of the right endpoint minus the left endpoint of I . Hence

$$m((a, b)) = m([a, b]) = m((a, b]) = m([a, b)) = b - a.$$

Lemma 6. Let $f : [0, 2] \rightarrow [0, 2]$ be the weather map, and let $I \subseteq [0, 2]$ be an interval. Then $f^n(I)$ is an interval for all $n \in \mathbb{N}$, and there exists $N \in \mathbb{N}$ such that $m(f^N(I)) \geq \min\{2m(I), 2\}$. Moreover, we can always choose $N \leq 4$.

PROOF: Since the weather map f is continuous, f^n is continuous for all $n \in \mathbb{N}$, and hence by the intermediate value theorem f^n takes connected sets to connected sets. Thus $f^n(I)$ is an interval for any interval $I \subseteq [0, 2]$.

Let I be an interval, and suppose that a is the left endpoint of I and b is the right endpoint of I . Then $m(I) = b - a$. Consider three cases.

CASE I: $b \leq 1$.

In this case $I \subseteq [0, 1]$, and since $f(x) = x + 1$ for all $x \in I$, it follows that $f(I)$ is an interval with left endpoint $a + 1$ and right endpoint $b + 1$. Thus $f(I) \subseteq [1, 2]$. Since $f(x) = 4 - 2x$ for all $x \in [1, 2]$, $f^2(I) = f(f(I))$ is an interval with left endpoint $2 - 2b$ and right endpoint $2 - 2a$. Thus

$$m(f^2(I)) = (2 - 2a) - (2 - 2b) = 2b - 2a = 2(b - a) = 2m(I) \geq \min\{2m(I), 2\}$$

and the claim holds with $N := 2$.

CASE II: $1 \leq a$.

In this case $I \subseteq [1, 2]$, and since $f(x) = 4 - 2x$ for all $x \in [1, 2]$, $f(I)$ is an interval with left endpoint $4 - 2b$ and right endpoint $4 - 2a$. Thus

$$m(f(I)) = (4 - 2a) - (4 - 2b) = 2b - 2a = 2(b - a) = 2m(I) \geq \min\{2m(I), 2\}$$

and the claim holds with $N := 2$.

CASE III: $a \leq 1$ and $1 \leq b$.

Since $m(I) = b - a = (b - 1) + (1 - a)$, either $b - 1 \geq m(I)/2$ or $1 - a \geq m(I)/2$. Consider the following three subcases.

SUBCASE III(i): $b - 1 \geq m(I)/2$.

Since $b \geq 1$ and $f(x) = 4 - 2x$ for all $x \geq 1$, we see that $f([1, b]) = (4 - 2b, 2]$. If $4 - 2b < 1$, then $f(4 - 2b, 2] = [0, 2]$. If $4 - 2b \geq 1$, then it follows from Case II that

$$m(f^2(4 - 2b, 2]) \geq \min\{2m((4 - 2b, 2]), 2\} = \min\{4b - 4, 2\}.$$

Thus in either situation, we have

$$\begin{aligned} m(f^2(4 - 2b, 2]) &\geq \min\{4b - 4, 2\} = \min\{4(b - 1), 2\} \\ &\geq \min\{4(m(I)/2), 2\} = \min\{2m(I), 2\} \end{aligned}$$

Hence

$$m(f^3(I)) \leq m(f^3([1, b]) = m(f^2((4 - 2b, 2])) \geq \min\{2m(I), 2\}$$

and the claim holds with $N := 3$.

SUBCASE III(ii): $1 - a \geq m(I)/2$ and $2 - 2a > 1$.

Since $a \leq 1$ and $f(x) = x + 1$ for all $x \leq 1$, we have $f((a, 1]) = (a + 1, 2]$ with $a + 1 \geq 1$. Since $f(x) = 4 - 2x$ for all $x \geq 1$, we have $f^2((a, 1]) = [0, 2 - 2a)$. Because $2 - 2a \geq 1$ in this case, it follows that $[0, 1] \subseteq f^2((a, 1])$ and hence

$$[0, 2] = f([1, 2]) = f^2([0, 1]) \subseteq f^2(f^2(a, 1)) = f^4((a, 1]) \subseteq f^4(I)$$

so that

$$m(f^4(I)) \geq m([0, 2]) = 2 \geq \min\{2m(I), 2\}$$

and the claim holds with $N := 4$.

SUBCASE III(iii): $1 - a \geq m(I)/2$ and $2 - 2a < 1$.

Since $a \leq 1$ and $f(x) = x + 1$ for all $x \leq 1$, we have $f((a, 1]) = (a + 1, 2]$ with $a + 1 \geq 1$. Since $f(x) = 4 - 2x$ for all $x \geq 1$, we have $f^2((a, 1]) = [0, 2 - 2a)$. Because $2 - 2a \leq 1$ in this case, it follows from Case I that $m(f^2([0, 2 - 2a))) \geq \min\{2m([0, 2 - 2a]), 2\} = \min\{4 - 4a, 2\}$. Hence

$$\begin{aligned} m(f^4(I)) &\geq m(f^4((a, 1])) = m(f^2(f^2((a, 1]))) = m(f^2([0, 2 - 2a))) \\ &\geq \min\{4 - 4a, 2\} = 4 - 4a = 4(1 - a) \geq 4(m(I)/2) \\ &= 2m(I) \geq \min\{2m(I), 2\} \end{aligned}$$

and the claim holds with $N := 4$.

QED

Lemma 7. *Let $f : [0, 2] \rightarrow [0, 2]$ be the weather map, and let $I \subseteq [0, 2]$ be an interval. Then $f^n(I)$ is an interval for all $n \in \mathbb{N}$, and for every $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $m(f^N(I)) \geq \min\{2^M m(I), 2\}$. Moreover, we can always choose $N \leq 4M$.*

PROOF: We proceed by induction on M . If $M = 1$, the result follows from Lemma 6. For the inductive step, let $M \in \mathbb{N}$ and suppose the claim holds for $M - 1$. Then there exists $N' \in \mathbb{N}$ with $m(f^{N'}(I)) \geq \min\{2^{M-1}m(I), 2\}$ and $N' \leq 4(M - 1)$. Applying Lemma 6 to the interval $f^{N'}(I)$, there exists $N'' \in \mathbb{N}$ with $m(f^{N''}(f^{N'}(I))) \geq \min\{2 \cdot 2^{M-1}m(I), 2\}$ and $N'' \leq 4$. If we let $N := N' + N''$, then $m(f^N(I)) \geq \min\{2 \cdot 2^{M-1}m(I), 2\} = \min\{2^M m(I), 2\}$ and $N = N' + N'' \leq 4(M - 1) + 4 = 4M$. QED

Lemma 8. *If U is a nonempty open subset of $[0, 2]$, and $m := m(U)$, then there exists $N \in \mathbb{N}$ such that $(0, 2) \subseteq f^N(U)$. Moreover, if U contains an interval I with $m(I) > 0$, then we may choose $N \leq 4 \log_2(2/m(I)) + 4$.*

PROOF: Since U is a nonempty open subset, then exists a nonempty open interval $I = (a, b)$ with $I \subseteq U$. Since $m(I) > 0$, we may choose a natural number M such that $\log_2(2/m(I)) \leq M < \log_2(2/m(I)) + 1$. By Lemma 7 there exists $N \leq 4M$ such that $f^N(I)$ is an interval and $m(f^N(I)) \geq \min\{2^M m(I), 2\}$. However,

$$2^M m(I) \geq 2^{\log_2(2/m(I))} m(I) = (2/m(I))m(I) = 2$$

so $m(f^N(I)) = 2$. Since $f^N(I)$ is an interval contained in $[0, 2]$ with length 2, it follows that $(0, 2) \subseteq f^N(I)$. Since $I \subseteq U$, we have $(0, 2) \subseteq f^N(I) \subseteq f^N(U)$. Moreover, $N \leq 4M < 4(\log_2(2/m(I)) + 1) = 4\log_2(2/m(I)) + 4$. QED

Theorem 7. *If $f : [0, 2] \rightarrow [0, 2]$ is the weather map, then f is topologically mixing and f satisfies the three properties in Devaney's definition of chaos (see Definition 5).*

PROOF: If U is a nonempty open set, it follows from Lemma 8 that there exists $N \in \mathbb{N}$ such that $(0, 2) \subseteq f^N(U)$. Thus $(0, 2) \subseteq f^n(U)$ for all $n \geq N$, and $f^n(U)$ intersects every nonempty open subset of $[0, 2]$ nontrivially. Thus f is topologically mixing. It follows that f is also topologically transitive, and Proposition 2 and Proposition 1 imply that f satisfies the three properties in Devaney's definition of chaos (cf. Remark 5). QED

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