CLASSIFICATION OF GRAPH C^*-ALGEBRAS AND LEAVITT PATH ALGEBRAS.

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Abstract. This is an extended abstract for my talk given at the Oberwolfach Mini-Workshop "Endomorphisms, Semigroups and C^*-algebras of Rings", held April 8–14, 2012 and organized by Joachim Cuntz, Wojciech Szymański, and Joachim Zacharias. This extended abstract will appear in the "Oberwolfach Report" (OWR) published by the Mathematisches Forschungsinstitut Oberwolfach.

In the past few years there have been a number of efforts to classify Leavitt path algebras associated with directed graphs. Many of these efforts have been modeled on the classification of C^*-algebras — particularly graph C^*-algebras and Cuntz-Krieger algebras — and there have been varying levels of success importing these techniques to the algebraic setting.

In the theory of C^*-algebras, Cuntz and Krieger introduced a class of C^*-algebras [5, 4] constructed from square matrices with entries in $\{0, 1\}$. The classification of Cuntz-Krieger algebras has been an ongoing project since their inception and great strides have been made, with the most general result recently obtained by Restorff in [9] where he classifies nonsimple Cuntz-Krieger algebras of matrices satisfying Condition (II) up to stable isomorphism via their filtrated K-theory.

Cuntz and Krieger showed that the structure and many of the invariants of $\mathcal{O}_A$ may be read off from A. If A is an $n \times n$ matrix with entries in $\{0, 1\}$, then $\mathcal{O}_{\mathcal{A}}$ is simple if and only if A is irreducible. In addition, $K_0(\mathcal{O}_A) \cong \mathbb{Z}^n/(I - A^t)\mathbb{Z}^n$ and $K_1(\mathcal{O}_{\mathcal{A}}) \cong \ker(I - A^t)$. In particular, one always has $K_0(\mathcal{O}_{\mathcal{A}}) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \mathbb{Z}^{n-k}$ and $\ker \mathcal{O}_A \cong \mathbb{Z}^{n-k}$, so that $K_1(\mathcal{O}_A)$ is equal to the free part of $K_1(\mathcal{O}_A)$, and all of the K-theory information is contained in K_0-group.

In classification of C^*-algebras one typically uses K-theory to obtain two classifications: A classification up to Morita equivalence; and a classification up to isomorphism. One of the first classifications up to isomorphism was obtained in 1981 by Enomoto, Fujii, and Watatani [7]. They showed that all simple Cuntz-Krieger algebras of 3×3 matrices are classified by $(K_0(\mathcal{O}_A), [1])$; i.e., the K_0-group together with the position of the unit. They accomplished this classification by describing moves on matrices that preserve the isomorphism class of the associated Cuntz-Krieger algebra, and showed that for any two such 3×3 matrices with the same invariant one may turn one matrix into the other via these moves.

In their original paper, Cuntz and Krieger observed that the Cuntz-Krieger algebras have an intimate relationship with symbolic dynamics, and they showed that if A is a square $\{0, 1\}$ matrix, and X_A is the two-sided shift space associated to A, then the dynamics of X_A is reflected in the structure of $\mathcal{O}_A$. In particular, they

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proved that if A and B are irreducible $\{0,1\}$ matrices and X_E is flow equivalent to X_F, then $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_B$. They gave two ways to prove this fact: (1) By realizing $\mathcal{O}_A \otimes \mathcal{K}$ as a crossed product; and (2) By using the fact flow equivalence is generated by the Strong Shift Equivalence moves plus the Parry-Sullivan move, and then showing that these matrix moves preserve the Morita equivalence class of the associated Cuntz-Krieger algebra.

In 1984 Franks proved his famous theorem in symbolic dynamics, which states that if A and B are irreducible square $\{0,1\}$ matrices, with $\operatorname{coker}(I-A) \cong \operatorname{coker}(I-A)$ and $\det(I-A) = \det(I-B)$, then X_A is flow equivalent to X_B. Since $K_0(\mathcal{O}_A) \cong \operatorname{coker}(I-A^t) \cong \operatorname{coker}(I-A)$ and $\det(I-A^t) = \det(I-A)$, we may combine this with Cuntz and Krieger's result to obtain the following fact: If A and B are irreducible with $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$ and $\det(I-A^t) = \det(I-B^t)$, then $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_B$. For a number of years it was wondered whether the determinant condition was necessary, and in 1995 Rørdam proved it is superfluous. In particular, if A is a matrix, we may form a new matrix A_- defined by

$$A_- := \begin{pmatrix} & & & 0 & 0 \\ & A & & \vdots & \vdots \\ & & & 1 & 0 \\ 0 & \cdots & 1 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

We call the process of going from A to A_- the "Cuntz splice". If $\mathcal{O}_2$ is the Cuntz algebra, and $\mathcal{O}_{2_-}$ is the corresponding Cuntz-Krieger algebra obtained by performing the Cuntz splice, then Cuntz proved that $\mathcal{O}_2 \cong \mathcal{O}_{2_-}$ implies that $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_{A_-}$ for all irreducible A. In 1995 Rørdam proved, using KK-theory, that indeed $\mathcal{O}_2 \cong \mathcal{O}_{2_-}$, and hence it is true that $\mathcal{O}_A \cong \mathcal{O}_{A_-}$ for all irreducible A [10]. Since the Cuntz splice changes the sign of the determinant (i.e., $\det(I-A^t) = -\det(I-A_-^t)$, it follows from Rørdam's result that if A and B are irreducible with $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$, then $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_B$ and one can turn A into B via a sequence of flow equivalence moves plus the Cuntz splice move. In addition, using a result of Huang, which states that an automorphism on the Bowen-Franks group is induced by a flow equivalence, Rørdam showed in [10] that if A and B are irreducible and $(K_0(\mathcal{O}_A), [1]) \cong (K_0(\mathcal{O}_B), [1])$, then $\mathcal{O}_A \cong \mathcal{O}_B$.

In 1997, generalizations of Cuntz-Krieger algebras, known as graph C^*-algebras were introduced. The Cuntz-Krieger algebras coincide with the C^*-algebras of finite directed graphs with no sinks or sources. As with the Cuntz-Krieger algebras, if E is a graph, then the structure of the graph C^*-algebra is reflected in E. In particular, if A_E is the vertex matrix of E, then $K_0(C^*(E)) \cong \operatorname{coker}(I - A_E^t)$ and $K_1(C^*(E)) \cong \ker(I - A_E^t)$. Unlike with the Cuntz-Krieger algebras, when the graph E is not finite, the K_1-group of $C^*(E)$ is not determined by the K_0-group of $C^*(E)$. The classification of Cuntz-Krieger algebras has been generalized to graph C^*-algebras in various ways, and more general C^*-algebra classification results have been applied to graph C^*-algebras as well [6]. In addition, there have been attempts to use the graph C^*-algebra classifications to obtain classifications for their algebraic counterparts, the Leavitt path algebras [3].

Purely algebraic classifications for the Leavitt path algebras were first initiated in the late 2000's. Interestingly, the development has been very similar to the classification of Cuntz-Krieger algebras. In 2008, Abrams, Ánh, Louly, and Pardo proved that $(K_0(L_K(E)), [1])$ is a complete isomorphism invariant for simple Leavitt path algebras of graphs with 3 vertices and no parallel edges. (This may be though of as an analogue of the result of Enomoto, Fujii, and Watatani for Cuntz-Krieger algebras of 3×3 matrices.) In addition, they proved this by exhibiting moves on the graphs that preserve the isomorphism class of the associated Leavitt path algebras.

In 2011, Abrams, Louly, Pardo, and Smith undertook a classification of Leavitt path algebras of finite graphs, in analogy with Rørdam's results for Cuntz-Krieger algebras. They were able to show that when the flow equivalence moves are performed on finite graphs, they preserve the Morita equivalence class of the associated Leavitt path algebra. Thus, using Franks' result, they prove that if E and F are finite graphs with no sinks and with simple Leavitt path algebras, then $K_0(L_K(E)) \cong K_0(L_K(F))$ and $\det(I - A_E^t) = \det(I - B_E^t)$ implies $L_K(E)$ is Morita equivalent to $L_K(F)$. Also, by applying Huang's result and using an argument similar to Rørdam's, they have shown that if, in addition, the isomorphism between K_0-groups preserves the class of the unit, then $L_K(E) \cong L_K(F)$. However, in all of the Leavitt path algebra results it is unknown if the determinant is necessary. As with the matrices of Cuntz-Krieger algebras, one can perform a Cuntz splice move to a graph. Abrams, Louly, Pardo, and Smith have shown that if $L_2 \cong L_{2_-}$ and this isomorphism lifts to an isomorphism of certain subalgebras of the endomorphism rings, then it follows that the Cuntz splice move preserves Morita equivalence for all finite graphs with no sinks that have simple Leavitt path algebras (see [2, Hypothesis on p.224] for a precise statement). Unfortunately, no one has been able to determine whether L_2 and L_{2_-} are isomorphic, much less whether there exists an isomorphism lifting to the subalgebras of the endomorphism rings. Thus it remains an open and important question as to whether the determinant is a Morita equivalence invariant for Leavitt path algebras, and in particular whether the Cuntz splice preserves Morita equivalence of the associated Leavitt path algebra.

Very recently, Sørensen has shown that if one uses the flow equivalence moves on graphs with finitely many vertices and infinitely many edges, then one does not need the Cuntz splice [11]. In particular, Sørensen has proven a Franks-type theorem that says the following: If E and F are each graphs with a finite number of vertices, an infinite number of edges, no sinks, and have associated graph C^*-algebras that are simple, and if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F)$, then there is a sequence of "flow equivalence moves" from E to F, each of which preserves Morita equivalence of the associated graph C^*-algebra. In particular, this implies that (unlike in the finite graph case) when there are an infinite number of edges, the Cuntz splice may be obtained by a sequence of "flow equivalence moves" on the graph. Using these results, Ruiz and the author have shown that the "flow equivalence moves" described by Sørensen also preserve Morita equivalence of the Leavitt path algebras, and if E and F are each graphs with a finite number of vertices, an infinite number of edges, no sinks, and their associated Leavitt path algebras are simple, and if $K_0(L_K(E)) \cong K_0(L_K(F))$ and $K_1(L_K(E)) \cong K_1(L_K(F)$, then there is a sequence of "flow equivalence moves" from E to F, and hence $L_K(E)$ and $L_K(F)$ are Morita equivalent. This shows that in this case the determinant is not a Morita equivalence invariant and the Cuntz splice is unnecessary in the classifying moves. It also implies that L_∞ is Morita equivalent to L_{∞_-}. In my talk I will discuss current efforts to determine whether L_2 and L_{2_-} are isomorphic or even Morita equivalent.

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