THE STRUCTURE OF GRAPH C*-ALGEBRAS AND THEIR GENERALIZATIONS

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These notes are an expanded version of the material covered by the author in his four talks at the Graph Algebra Workshop in Málaga, Spain during July 3–8, 2006. These four talks were given on Tuesday, July 4 following Iain Raeburn's lectures on Monday, and throughout these notes we will assume familiarity with some of the basic material he covered (much of which can be found in Chapters 1 and 2 of [34]). Our goal in these notes is to provide self-contained proofs of some of the results concerning ideal structure of graph algebras, and also to survey certain additional topics such as desingularization, K-theory and its applications to classifying C^*-algebras, and various generalizations of graph algebras.

In these notes we will follow the convention of having the partial isometries in a graph algebra go in a direction opposite the edge (so the source projection of s_e is $p_{r(e)}$ and the range projection of s_e is dominated by $p_{s(e)}$). This is the convention used in most of the graph C^*-algebra literature. However, it is not the convention recently adopted by Raeburn in his notes from this workshop and in his book [34]. Nonetheless, the author feels there are several good reasons for breaking from the convention used by Raeburn and instead have the edges go the "classic" direction. In the author's opinion, much of the notation and many results in the subject take a more natural form when one has the edges going this way; and furthermore, much of the notation agrees with notation and conventions from other subjects. A few examples are:

(1) With our convention, graph properties are often stated in terms of traversing paths *forward* and being able to reach certain vertices. For example: cofinality means that any vertex can reach any infinite path by following edges forward; we will frequently talk of vertices being able to reach loops by following edges forward; and a set H is said to be hereditary if, when following edges forward, once one enters H one stays in H. If one uses the alternate convention, one must instead rephrase all these results in terms of "inverse reaching" or following directed edges backwards.

Date: December 21, 2006.

2000 *Mathematics Subject Classification.* 46L55.

Key words and phrases. C^*-algebra, graph algebra, ideal structure, desingularization, K-theory, ultragraph, topological quiver, Cuntz-Pimsner algebra, C^*-correspondence.

(2) When we write a path $e_1 \dots e_n$ we have $r(e_i) = s(e_{i+1})$, so that the path traverses edges in the same way one reads them: from left to right. Also the source of this path is $s(e_1)$ and the range is $r(e_n)$, and when we speak of infinite paths they are of the form $e_1 \dots$ starting at $s(e_1)$. If one uses the alternate convention, then a path $e_1 \dots e_n$ has source $s(e_n)$ and range $r(e_1)$ and one traverses the path from right to left. Also, in the alternate convention infinite paths are of the form $e_1 e_2 \dots$ ending at $r(e_1)$.

(3) If we want to realize the AF-algebra with Bratteli diagram E as a full corner of the graph algebra $C^*(E)$, then our convention agrees with the conventions used in Bratteli diagrams. With the alternate convention, one has to reverse the edges of the Bratteli diagram. (See [34, p.20–21] for more details.)

(4) Our convention agrees with the conventions used in Leavitt Path Algebras (which are based off of graph conventions in Algebra). In particular, with our convention, the partial isometries satisfy the same relations as the generators of the Leavitt Path Algebra. Since Leavitt Path Algebras have a great deal in common with graph C^*-algebras, this allows one to more easily compare results for the two objects.

(5) With our convention, if A is the vertex matrix of a graph then $A(v, w)$ is the number of edges from v to w. Again, this agrees with reading from left to right, and it also agrees with the convention used in graph theory. If one uses the alternate convention, then $A(v, w)$ is the number of edges from w to v, which forces one to read from right to left, and does not agree with the matrix used by graph theorists. Similarly for the edge matrix B; in our convention $B(e, f) = 1$ if and only if $r(e) = s(f)$. In the alternate convention, $B(e, f) = 1$ if and only if $r(f) = s(e)$.

(6) In most of the literature — particularly in many of the seminal papers on graph C^*-algebras — our convention has been used. If one is first learning the subject, or if one needs to refer to these papers frequently, it is much easier to use this convention. Of course, one can always argue that a person simply needs to "reverse the edges" when reading these papers. But, this is often trickier than it sounds, and the author (who tends to be right/left challenged himself at times) wants to make the literature and its results as accessible as possible to the non-expert.

Remark. While we use the conventions that our partial isometries go in a direction opposite our edges, it is certainly true that for higher-rank graphs it is useful to have the partial isometries go in the same direction as the edges. This is because it is more natural categorically; in fact, in (2) above one sees that edges in a path are "composed" in the same way as morphisms — from right to left. However, despite the fact that ordinary graphs are rank 1 graphs, it does not seem that this is sufficient reason for using the higher-rank graph convention in this setting. (Regardless of what those working in higher rank graphs may tell you!) Because these categorical considerations are less important in the rank 1 graph setting, and because many of the advantages of using the higher rank convention disappear or become marginal in this special case, it seems that in light of the reasons in (1)–(6) above it makes sense to have separate conventions for higher rank graphs and for ordinary directed graphs.

We now establish some notation and terminology that we shall use frequently. A directed graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. Since all our graphs will be directed, we will often simply call a directed graph a "graph". A vertex $v \in E^0$ is called a *sink* if $|s^{-1}(v)| = 0$, and v is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. If v is either a sink or an infinite emitter, then we call v a *singular vertex.* If v is neither a sink nor an infinite emitter, then we say v is a *regular vertex.* A graph is said to be *row-finite* if it has no infinite emitters. (Note that row-finite graphs are allowed to have sinks.)

If E is a graph we define a *Cuntz-Krieger E-family* to be a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges which satisfy the *Cuntz-Krieger relations*:

(1) $s_e^* s_e = p_{r(e)}$ for every $e \in E^1$;
(2) $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$;
(3) $p_v = \sum_{s(e)=v} s_e s_e^*$ for every $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$.

The *graph C^*-algebra* $C^*(E)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger E-family. We sometimes refer to the graph C^*-algebra as simply the graph algebra.

A *path* in E is a sequence of edges $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$, and we say that α has length $|\alpha| = n$. We let E^n denote the set of all paths of length n, and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of finite paths in E. Note that vertices are considered paths of length zero. The maps r and s extend to E^*, and for $v, w \in G^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. It is a consequence of the Cuntz-Krieger relations that $C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$.

We say that a path $\alpha := \alpha_1 \ldots \alpha_n$ of length 1 or greater is a *loop* if $r(\alpha) = s(\alpha)$, and we call the vertex $s(\alpha) = r(\alpha)$ the *base point* of the loop. An *exit* for a loop $\alpha_1 \ldots \alpha_n$ is an edge $f \in E^1$ with the property that $s(f) = s(\alpha_i)$ but $\alpha_i \neq f$ from some $i \in \{1, \ldots n\}$. We say that a graph satisfies Condition (L) if every loop in the graph has an exit.

By an *ideal* in a C^*-algebra A we will mean a closed, two-sided ideal in A. If E is a graph, then by the universal property of $C^*(E)$ there exists a *gauge action* $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ with the property that $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$ for all $z \in \mathbb{T}$. If $\gamma : \mathbb{T} \to \operatorname{Aut} A$ is this gauge action, then we say an ideal I in A is *gauge-invariant* if $\gamma_z(a) \in I$ for all $a \in I$ and $z \in \mathbb{T}$.

1. Simplicity and Ideal Structure

In this section we shall use the uniqueness theorems to analyze the structure of ideals in a graph C^*-algebra and give conditions for simplicity.

Theorem 1.1 (Gauge-Invariant Uniqueness Theorem)**.** *Let $E = (E^0, E^1, r, s)$ be a directed graph and let $\rho : C^*(E) \to B$ be a $*$-homomorphism from $C^*(E)$ into a C^*-algebra B. Also let γ denote the standard gauge action on $C^*(E)$. If there exists an action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for each $z \in \mathbb{T}$, and if $\rho(p_v) \neq 0$ for all $v \in E^0$, then ρ is injective.*

Note that the condition $\beta_z \circ \rho = \rho \circ \gamma_z$ for each $z \in \mathbb{T}$ is sometimes summarized by saying that ρ is *equivariant* for the gauge actions β and γ.

Theorem 1.2 (Cuntz-Krieger Uniqueness Theorem)**.** *Let $E = (E^0, E^1, r, s)$ be a directed graph satisfying Condition (L) and let $\rho : C^*(E) \to B$ be a $*$-homomorphism from $C^*(E)$ into a C^*-algebra B. If $\rho(p_v) \neq 0$ for all $v \in E^0$, then ρ is injective.*

Although both of these uniqueness theorems hold for arbitrary graphs, to simplify our analysis in this section we shall only consider C^*-algebras of row-finite graphs. We will discuss the general (non-row-finite) case in Section 2

Our analysis in this section will proceed in the following stages:

- First, we will use the Gauge-Invariant Uniqueness Theorem to classify the gauge-invariant ideals of $C^*(E)$. This will consist of showing the following three facts:
  1. The gauge-invariant ideals of $C^*(E)$ correspond to saturated hereditary subsets of vertices of E^0.
  2. If I_H is the gauge-invariant ideal corresponding to the saturated hereditary subset H, then I_H is Morita equivalent to the C^*-algebra of the subgraph of E whose vertices are H and whose edges are the edges of E whose source is a vertex in H.
  3. If I_H is the gauge-invariant ideal corresponding to the saturated hereditary subset H, then the quotient $C^*(E)/I_H$ is isomorphic to the C^*-algebra of the subgraph of E whose vertices are $E^0 \setminus H$ and whose edges are the edges of E whose range is a vertex in $E^0 \setminus H$.
- Next we shall derive a condition, called Condition (K), which is equivalent to having all ideals of $C^*(E)$ be gauge-invariant. Our classification of gauge-invariant ideals then gives a complete description of the ideals of a C^*-algebra associated to a graph satisfying Condition (K).
- Finally we shall obtain conditions for $C^*(E)$ to be simple. We will give various equivalent forms for these conditions.

As we work to prove these facts we will use the following definitions.

Definition 1.3. Let $E = (E^0, E^1, r, s)$ be a graph. A subset $H \subseteq E^0$ is *hereditary* if for any $e \in E^1$ we have $s(e) \in H$ implies $r(e) \in H$. A hereditary subset $H \subseteq E^0$ is said to be *saturated* if whenever $v \in E^0$ is a regular vertex with $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$, then $v \in H$. If $H \subseteq E^0$ is a hereditary set, the *saturation of* H is the smallest saturated subset $\overline{H}$ of E^0 containing H.

Roughly speaking, a subset of vertices is hereditary if no vertex in H points outside of H. This set H is also saturated if whenever a regular vertex points only into H then that vertex is in H.

Example 1.4. In the graph

$$\begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[r] \arrow[dr, bend left] & w \\ x & y \arrow[u] \arrow[r] & z \arrow[u] \arrow[ul, bend left] \end{tikzcd}$$

the set $X = \{v, w, z\}$ is hereditary but not saturated. However, the set $H = \{v, w, y, z\}$ is both saturated and hereditary. We see that $\overline{X} = H$.

For any graph, the saturated hereditary subsets of vertices form a lattice with the ordering given by set inclusion, the infimum given by $H_1 \wedge H_2 := H_1 \cap H_2$, and the supremum given by $H_1 \vee H_2 := \overline{H_1 \cup H_2}$.

Definition 1.5. For $v, w \in E^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. In this case we say that v *can reach* w.

Note that of H is a hereditary subset and $v \geq w$ with $v \in H$, then $w \in H$.

1.1. **Classification of Gauge-Invariant Ideals.** We wish to prove the following theorem. Our approach will be similar to the proof of [2, Theorem 4.1].

Theorem 1.6. *Let $E = (E^0, E^1, r, s)$ be a row-finite graph. For each subset $H \subseteq E^0$ let I_H denote the ideal in $C^*(E)$ generated by $\{p_v : v \in H\}$.*

(a) *The map $H \mapsto I_H$ is an isomorphism from the lattice of saturated hereditary subsets of E onto the lattice of gauge-invariant ideals of $C^*(E)$.*

(b) *If H is a saturated hereditary subset of E^0, and we let $E \setminus H$ be the subgraph of E whose vertices are $E^0 \setminus H$ and whose edges are $E^1 \setminus r^{-1}(H)$, then $C^*(E)/I_H$ is canonically isomorphic to $C^*(E \setminus H)$.*

(c) *If X is any hereditary subset of E^0, then $I_X = I_{\overline{X}}$. Furthermore, if we let E_X denote the subgraph of E whose vertices are X and whose edges are $s^{-1}(X)$, then $C^*(E_X)$ is canonically isomorphic to the subalgebra $C^*(\{s_e, p_v : e \in s^{-1}(X) \text{ and } v \in X\})$, and this subalgebra is a full corner of the ideal I_X.*

Remark 1.7. Observe that in (b) the fact that H is hereditary implies that if $e \in E^1 \setminus r^{-1}(H)$, then $s(e) \in E^0 \setminus H$. Likewise in (c) the fact that X is hereditary implies that if $e \in s^{-1}(X)$, then $r(e) \in X$.

Example 1.8. If E is the graph

$$\begin{tikzcd}
u \arrow[loop above] \arrow[loop left] \arrow[dr] & & & \\
& v \arrow[r] & w \arrow[r] & x \arrow[loop above] \arrow[loop below] \\
y \arrow[loop left] \arrow[loop below] \arrow[ur] & & &
\end{tikzcd}$$

then the saturated hereditary subsets of E are: $\emptyset$, $\{v, w, x\}$, $\{u, v, w, x\}$, $\{v, w, x, y\}$, and $E^0 = \{u, v, w, x, y\}$. When these subsets are ordered by inclusion we have the following lattice

$$\begin{tikzcd}[arrows=-]
& E^0 & \\
\{u,v,w,x\} \arrow[ur] & & \{v,w,x,y\} \arrow[ul] \\
& \{v,w,x\} \arrow[ul] \arrow[ur] & \\
& \emptyset \arrow[u] &
\end{tikzcd}$$

and by Part (a) of Theorem 1.6 the lattice of gauge-invariant ideals in $C^*(E)$ is

$$\begin{tikzcd}[arrows=-]
& I_{E^0} = C^*(E) & \\
I_{\{u,v,w,x\}} \arrow[ur] & & I_{\{v,w,x,y\}} \arrow[ul] \\
& I_{\{v,w,x\}} \arrow[ul] \arrow[ur] & \\
& I_\emptyset = \{0\} \arrow[u] &
\end{tikzcd}$$

Hence $C^*(E)$ has three proper nontrivial gauge-invariant ideals. If we let $H = \{v, w, x\}$, then $E \setminus H$ and E_H are the following graphs

$$E \setminus H \qquad \begin{tikzcd} u \arrow[loop above] \arrow[loop below] \\ y \arrow[loop above] \arrow[loop below] \end{tikzcd} \qquad\qquad E_H \qquad \begin{tikzcd} v \arrow[r] & w \arrow[r] & x \arrow[loop above] \arrow[loop below] \end{tikzcd}$$

and by Parts (b) and (c) of Theorem 1.6 we have $C^*(E)/I_H \cong C^*(E \setminus H) \cong \mathcal{O}_2 \oplus \mathcal{O}_2$ and $C^*(E_H)$ is a full corner (and hence Morita equivalent) to I_H.

In addition, if we let $X = \{x\}$, then X is hereditary (but not saturated) and $\overline{X} = H$. We see that the graph E_X is

$$E_X \qquad \begin{tikzcd} x \arrow[loop above] \arrow[loop below] \end{tikzcd}$$

Part (c) of Theorem 1.6 tells us that $C^*(E_X) \cong \mathcal{O}_2$ is also a full corner of the ideal $I_X = I_{\overline{X}} = I_H$. Thus $C^*(E_H)$ and $C^*(E_X)$ are Morita equivalent. However, the C^*-algebras $C^*(E_H)$ and $C^*(E_X)$ are not isomorphic — with a little bit of work one can show that $C^*(E_H) \cong M_3(\mathcal{O}_2)$.

Before we can provide a proof of Theorem 1.6 we will need a few lemmas.

Lemma 1.9. *Let E be a graph, and let I be an ideal in $C^*(E)$. Then $H := \{v \in E^0 : p_v \in I\}$ is a saturated hereditary subset of E^0.*

Proof. Suppose $e \in E^1$ with $s(e) \in H$. Then $p_{s(e)} \in I$, and because I is an ideal we have $p_{r(e)} = s_e^* s_e = s_e^* p_{s(e)} s_e \in I$. Hence $r(e) \in H$ and H is hereditary.

Next suppose $v \in E^0$ is a regular vertex and $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$. Then $p_{r(e)} \in I$ for every $e \in s^{-1}(v)$, and since I is an ideal $s_e s_e^* = s_e p_{r(e)} s_e^* \in I$ for every $e \in s^{-1}(v)$. Because v is a regular vertex we have that $p_v = \sum_{s(e)=v} s_e s_e^* \in I$. Thus $v \in H$ and H is saturated. $\square$

Remark 1.10. Notice that in order to prove H is saturated in the above lemma, we needed to have the relation $p_v = \sum_{s(e)=v} s_e s_e^*$. This is why the definition of saturated only requires that $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$ implies $v \in H$ when v is a regular vertex.

Lemma 1.11. *Let E be a graph, and let X be a hereditary subset of E^0. Then*

$$I_X = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \ : \ \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{X}\}. \tag{1.1}$$

In particular, this implies that $I_X = I_{\overline{X}}$ and that I_X is gauge invariant.

Proof. We first note that it follows from Lemma 1.9 that $\{v \in E^0 : p_v \in I_X\}$ is a saturated set containing X, and thus containing $\overline{X}$. Thus the right hand side of (1.1) is contained in I_X. Furthermore, any non-zero product of the form $(s_\alpha s_\beta^*)(s_\gamma s_\delta^*)$ collapses to a term of the form $s_\mu s_\nu^*$, and by examining the various possibilities μ and ν, and using the hereditary property of $\overline{X}$ we deduce that the right hand side of (1.1) is an ideal. Since the right hand side of (1.1) contains the generators of I_X, the equality in (1.1) holds. $\square$

Lemma 1.12. *Let E be a graph and let H be a saturated hereditary subset of E^0. If I_H is the ideal in $C^*(E)$ generated by $\{p_v : v \in H\}$, then $\{v \in E^0 : p_v \in I_H\} = H$.*

Proof. We trivially have that $v \in H$ implies $p_v \in I_H$, so $H \subseteq \{v \in E^0 : p_v \in I_H\}$. For the reverse inclusion, choose a Cuntz-Krieger $(E \setminus H)$-family $\{S_e, P_v : e \in (E \setminus H)^1, v \in (E \setminus H)^0\}$ that generates $C^*(E \setminus H)$. We may extend this to a Cuntz-Krieger E-family by setting $P_v = 0$ when $v \in H$ and $S_e = 0$ when $r(e) \in H$. To see that this is a Cuntz-Krieger E-family notice that H hereditary implies the Cuntz-Krieger relations holds at vertices in H, and H saturated implies there are no vertices in $(E \setminus H)^0 = E^0 \setminus H$ at which a new Cuntz-Krieger relation is being imposed (in other words, all sinks of $E \setminus H$ are sinks in E). The universal property of $C^*(E)$ then gives a homomorphism $\rho : C^*(E) \to C^*(\{S_e, P_v\})$ which vanishes on I_H since it kills all the generators $\{p_v : v \in H\}$. But $\rho(p_v) = P_v \neq 0$ for $v \notin H$, so $v \notin H$ implies $p_v \notin I_H$. Thus $\{v \in E^0 : p_v \in I_H\} \subseteq H$. $\square$

Lemma 1.13. *Let E be a graph and let X be any subset of E^0. Then there exists a projection $p_X \in M(C^*(E))$ such that*

$$p_X s_\alpha s_\beta^* = \begin{cases} s_\alpha s_\beta^* & \text{if } s(\alpha) \in X \\ 0 & \text{if } s(\alpha) \notin X \end{cases}.$$

Proof. If X is finite, then the projection $p_X := \sum_{v \in X} p_v$ has the required properties. Therefore, we need only consider the case when X is infinite. If X is infinite list the elements of X as $X = \{v_1, v_2, \ldots\}$. For each $N \in \mathbb{N}$ let $p_N := \sum_{n=1}^N p_{v_n}$. Then

$$p_N s_\alpha s_\beta^* = \begin{cases} s_\alpha s_\beta^* & \text{if } s(\alpha) = v_n \text{ for some } n \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Thus for any $a \in \operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$, the sequence $\{p_N a\}_{N=1}^\infty$ is eventually constant. An $\epsilon/3$ argument then shows that $\{p_N a\}_{N=1}^\infty$ is Cauchy for every $a \in C^*(E)$. Thus we may define $p : A \to A$ by $p(a) = \lim_{N \to \infty} p_N a$. Since

$$\langle b, p(a) \rangle = b^* p(a) = \lim_{N \to \infty} p_N a = \lim_{N \to \infty} (p_N b)^* a = p(b)^* a = \langle p(b), a \rangle$$

we see that the map p is an adjointable operator on the Hilbert C^*-module A_A with $p^* = p$. Consequently we have defined a multiplier p of A [36, Theorem 2.47] satisfying the required equalities. Finally, we see that

$$p^2(a) = p(\lim_N p_N a) = \lim_M p_M(\lim_N p_N a) = \lim_M(\lim_N p_M p_N a) = \lim_M p_M a = p(a)$$

so that $p^2 = p$, and p is a projection. □

We will now prove the various parts of Theorem 1.6. We will find it convenient to first prove Part (b), and then to prove Part (a) and Part (c).

Proof of Theorem 1.6(b). Let H be a saturated hereditary subset of H. If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family generating $C^*(E)$, then the collection $\{s_e + I_H, p_v + I_H : e \in (E \setminus H)^0, v \in (E \setminus H)^1\}$ in $C^*(E)/I_H$ is a Cuntz-Krieger $(E \setminus H)$-family. — The first two Cuntz-Krieger relations are immediate. To see the third, notice that if $e \in E^1$ with $r(e) \in H$, then $p_{r(e)} \in I_H$ and $s_e = s_e p_{r(e)} \in I_H$, so $s_e + I_H = 0 + I_H$ and

$$\begin{aligned} p_v + I_H &= \left(\sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^* \right) + I_H \\ &= \sum_{\{e \in E^1 \setminus r^{-1}(H) : s(e) = v\}} (s_e + I_H)(s_e + I_H)^* \\ &\qquad\qquad + \sum_{\{e \in r^{-1}(H) : s(e) = v\}} (s_e + I_H)(s_e + I_H)^* \\ &= \sum_{\{e \in (E \setminus H)^1 : s(e) = v\}} (s_e + I_H)(s_e + I_H)^*. \end{aligned}$$

By the universal property of $C^*(E \setminus H)$ there is a homomorphism $\rho : C^*(E \setminus H) \to C^*(E)/I_H$ taking the generators of $C^*(E \setminus H)$ canonically to the elements of $\{s_e + I_H, p_v + I_H : e \in (E \setminus H)^0, v \in (E \setminus H)^1\}$. Since I_H is gauge invariant by Lemma 1.11, the gauge action on $C^*(E)$ descends to a gauge action on $C^*(E)/I_H$, and by checking on generators it is straightforward to verify that ρ is equivariant for the gauge actions on $C^*(E/H)$ and $C^*(E)/I_H$. Furthermore, since H is saturated and hereditary Lemma 1.12 implies that $p_v \notin I_H$ when $v \notin H$, and thus $\rho(p_v) = p_v + I_H \neq 0$ when $v \in (E \setminus H)^0$. It then follows from the Gauge-Invariant Uniqueness Theorem that ρ is injective. In addition, we know that the elements of $\{p_v + I_H, s_e + I_H : v \in E^0, e \in E^1\}$ generate $C^*(E)/I_H$, and because $p_v + I_H = 0 + I_H$ when $v \in H$ and $s_e + I_H = 0 + I_H$ when $r(e) \in H$, we have that the elements $\{p_v + I_H, s_e + I_H : v \in (E \setminus H)^0, e \in (E \setminus H)^1\}$ generate $C^*(E)/I_H$. Thus ρ is surjective, and an isomorphism. □

Proof of Theorem 1.6(a). It follows from Lemma 1.11 that the mapping $H \mapsto I_H$ maps from the lattice of saturated hereditary subsets of E^0 into the lattice of gauge-invariant ideals of $C^*(E)$. We shall show that this mapping is surjective. Let I be a gauge-invariant ideal in $C^*(E)$, and set $H := \{v \in E^0 : p_v \in I\}$. It follows from Lemma 1.9 that H is saturated and hereditary. Since $I_H \subseteq I$, we see that $p_v \notin I$ implies $p_v \notin I_H$. Hence

I and I_H contain exactly the same set of projections $\{p_v : v \in H\}$. Also, because $I_H \subseteq I$ we may define a quotient map $q : C^*(E)/I_H \to C^*(E)/I$ by $q(a + I_H) = a + I$. (Strictly speaking, q is simply the quotient map from $C^*(E)/I_H$ onto $(C^*(E)/I_H)/(I/I_H)$.) Theorem 1.6(b) implies that there is a canonical isomorphism $\rho : C^*(E \setminus H) \to C^*(E)/I_H$. If we consider the composition $q \circ \rho : C^*(E \setminus H) \to C^*(E)/I$, then because ρ is canonical, and because I and I_H contain the same set of projections $\{p_v : v \in H\}$, it follows that $q \circ \rho$ is nonzero on the generating projections of $C^*(E \setminus H)$. Furthermore, since I is gauge invariant, the gauge action on $C^*(E)$ descends to a gauge action on $C^*(E)/I$ and by checking on generators (and once again using the fact that ρ is canonical) we can verify that $q \circ \rho$ is equivariant for the gauge actions on $C^*(E \setminus H)$ and $C^*(E)/I$. The Gauge-Invariant Uniqueness Theorem then implies that $\rho \circ q$ is injective. Therefore q is injective, and since $q : C^*(E)/I_H \to C^*(E)/I$ is the quotient map, this implies that $I = I_H$. Hence the mapping $H \mapsto I_H$ is surjective.

Next we shall show that the map $H \mapsto I_H$ is injective. If H and K are saturated hereditary subsets with $I_H = I_K$, then $\{v \in E^0 : p_v \in I_H\} = \{v \in E^0 : p_v \in I_K\}$ and Lemma 1.12 implies that $H = K$.

Finally, we need to show that the map $H \mapsto I_H$ is a lattice isomorphism. Since $H \subset K$ implies that $I_H \subseteq I_K$, we see that the map preserves the order structure of the lattices. Because the map is also a bijection, this implies that it is a lattice isomorphism. □

Proof of Theorem 1.6(c). Fix a hereditary subset X of E, and let p_X be the projection in $M(C^*(E))$ defined in Lemma 1.13. The fact that $I_X = I_{\overline{X}}$ follows from Lemma 1.11. Furthermore, Lemma 1.11 implies that $I_X = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \ : \ \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{X}\}$. Because X is hereditary, the elements $\{s_e, p_v : e \in s^{-1}(X), v \in X\}$ forms a Cuntz-Krieger E_X-family. (In particular, to get the third Cuntz-Krieger relation we use the fact that X is hereditary to conclude that $p_{r(e)}$ is in this set whenever $p_{s(e)}$ is in the set.) By the universal property of $C^*(E_X)$ there exists a surjective homomorphism $\rho : C^*(E_X) \to C^*(\{s_e, p_v : e \in s^{-1}(X), v \in X\})$, and since the gauge action on $C^*(E)$ restricts to a gauge action on $C^*(\{s_e, p_v : e \in s^{-1}(X), v \in X\})$, an application of the Gauge-Invariant Uniqueness Theorem shows that ρ is an isomorphism.

Furthermore, since compression by the projection p_X is linear and continuous, and since X is hereditary, we have that

$$\begin{aligned} p_X I_X p_X &= \overline{\operatorname{span}}\{p_X s_\alpha s_\beta^* p_X \ : \ \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{X}\} \\ &= \overline{\operatorname{span}}\{s_\alpha s_\beta^* \ : \ \alpha, \beta \in E^*, s(\alpha) \in X, s(\beta) \in X, \text{ and } r(\alpha) = r(\beta)\} \\ &= C^*(E_X). \end{aligned}$$

Finally, we see that the corner $p_X I_X p_X$ is full since $\{p_v : v \in X\}$ generates I_X. □

1.2. **Condition (K).** In the previous section we described and analyzed the structure of the gauge-invariant ideals in a graph algebra. However, typically a graph algebra will have many ideals besides the gauge-invariant ones. In this section we shall derive a condition on a graph, called Condition (K), that will ensure all ideals in the associated C^*-algebra are gauge invariant. Thus for C^*-algebras of row-finite graphs satisfying Condition (K), Theorem 1.6 gives a complete description of the ideals.

If E is a row-finite graph, and I is an arbitrary ideal in $C^*(E)$, then we must ask: "What conditions on E would require that I be gauge invariant?" Theorem 1.6(a) shows that any gauge-invariant ideal is of the form I_H, and therefore is generated by the p_v's which it contains. So we are really trying to show that given an ideal I we can recover it as $I = I_H$ for $H = \{v \in E^0 : p_v \in I\}$.

This is reminiscent of what we had to do when we proved that the map $H \mapsto I_H$ is surjective in the first paragraph of the proof of Theorem 1.6(a). There we created a map $q \circ \rho : C^*(E \setminus H) \to C^*(E)/I$, and used the Gauge-Invariant Uniqueness Theorem to conclude that this map was injective and $I = I_H$. But what if we do not know a priori that I is gauge invariant? We can still create the map $q \circ \rho : C^*(E \setminus H) \to C^*(E)/I$, but we will not be able to apply the Gauge-Invariant Uniqueness Theorem because we do not know that $C^*(E)/I$ has the necessary gauge action. However, not all is lost — we could instead apply our other uniqueness theorem: The Cuntz-Krieger Uniqueness Theorem. We will not be able to do this in general, however; in order to apply the Cuntz-Krieger Uniqueness Theorem we need to know that the subgraph $E \setminus H$ satisfies Condition (L).

This is exactly the condition we want to ensure that all ideals are gauge invariant: For any saturated hereditary set E the subgraph $E \setminus H$ satisfies Condition (L). However, because this is not a condition that is easy to check by quickly looking at a graph, we will give a different formulation of this condition in terms of "simple loops", and then prove the two notions are equivalent.

Definition 1.14. A *simple loop* in a graph E is a loop $\alpha \in E^*$ with the property that $s(\alpha_i) \neq s(\alpha_1)$ for $i \in \{2, 3, \dots, |\alpha|\}$.

In particular, a simple loop is allowed to repeat vertices or edges as it traverses through the graph, provided that it returns to the base point only at the end of its journey and not before.

Definition 1.15. A graph E is said to satisfy *Condition (K)* if no vertex in E is the base point of exactly one simple loop; that is, every vertex in E is either the base point of no loops or of more than one simple loop.

Beware the subtleties of Condition (K)! It is not uncommon for someone who first encounters this definition to think they understand Condition (K) only to come across an example at a later time that causes confusion. For example, the graph

$$\begin{tikzcd} v \arrow[r, bend left, "e"] & w \arrow[l, bend left, "g"] \arrow[loop right, "f"] \end{tikzcd}$$

satisfies Condition (K) because eg and efg are two simple loops based at v, and f and ge are two simple loops based at w. (There are, of course, many other simple loops besides the ones we mentioned. For example, $effg$ is also a simple loop based at v.)

Likewise, the graph

$$\begin{tikzcd} v \arrow[r, bend left] \arrow[r, bend right] & w \arrow[r] \arrow[d] & x \\ & y \arrow[ul] & \end{tikzcd}$$

satisfies Condition (K), because there are no loops based at x and every other vertex is the base point of at least two simple loops.

Remark 1.16. Notice that Condition (K) implies Condition (L). To see this, let E be a graph satisfying condition (K). If α is a loop in E, then $v = s(\alpha)$ is the base point of a loop, and hence there is at least one simple loop based at v. But then α must have an exit, for otherwise there would be a unique simple loop based at v.

Proposition 1.17. *If E is a graph, then E satisfies Condition (K) if and only if for every saturated hereditary subset H of E^0 the subgraph $E \setminus H$ satisfies Condition (L).*

Proof. Suppose E satisfies Condition (K). If H is a saturated hereditary subset of E^0, and α is a loop in $E \setminus H$, then $v = s(\alpha)$ is a vertex in $E^0 \setminus H$. Since α is also a loop in E, there must exist a second loop β in E based at v. Since $s(\beta) \notin H$, and since H is hereditary, it follows that each of the elements of $\{r(\beta_i)\}_{i=1}^{|\beta|}$ is an element of $E^0 \setminus H$. Thus the edges $\{\beta_i\}_{i=1}^{|\beta|}$ are elements of $(E \setminus H)^1 = E^1 \setminus r^{-1}(H)$, and β is a loop in $E \setminus H$ based at v. Since there are two distinct loops in $E \setminus H$ based at v, it follows that α has an exit in $E \setminus H$.

Conversely, suppose that $E \setminus H$ satisfies Condition (L) for every saturated hereditary subset H of E^0. Let v be a vertex, and let α be a simple loop based at v. Define $H := \{w \in E^0 : w \not\geq v\}$. It is straightforward to verify that H is hereditary, and since v is on a loop H is also saturated. Because the vertices on α can all reach v, α is a loop in $E \setminus H$. By hypothesis α has an exit $e \in (E \setminus H)^1$. Suppose that $s(e) = s(\alpha_k)$ for some $k \in \{1, 2, \ldots, |\alpha|\}$. Since $r(e) \in (E \setminus H)^0 = E^0 \setminus H$, we have that $r(e) \notin H$ and $r(e) \geq v$. Thus there exists a path $\mu \in E^*$ with $s(\mu) = r(e)$ and $r(\mu) = v$, and furthermore we may choose the path μ so that $r(\mu_i) \neq v$ for $1 \leq i < |\mu|$. But then $\beta := \alpha_1 \ldots \alpha_k e \mu$ is a simple loop based at v, which is distinct from α. In addition, since the vertices on β can reach v, it follows that β is a loop in $E \setminus H$. Hence $E \setminus H$ satisfies Condition (K). $\square$

We shall now show that Condition (K) characterizes those row-finite graphs having C^*-algebras whose ideals are all gauge-invariant. To do this we will first need a lemma showing that if E is a graph containing a loop with no exits, then $C^*(E)$ has an ideal that is not gauge invariant — in fact, we will show there are many ideals in $C^*(E)$ that are not gauge invariant.

Lemma 1.18. *If E is a row-finite graph containing a loop with no exits, then $C^*(E)$ contains an uncountable number of ideals that are not gauge invariant.*

Proof. Let α be a loop in E that has no exits, and let $X = \{s(\alpha_i)\}_{i=1}^{|\alpha|}$. Then since α has no exits we see that for all $i \in \{1, 2, \ldots, |\alpha|\}$ it is the case that α_i is the only edge whose source is $s(\alpha_i)$. Thus X is hereditary. If we let $p_X := \sum_{i=1}^{|\alpha|} p_{s(\alpha_i)}$, then as shown in the proof of Theorem 1.6(c), we have that $C^*(E_X)$ is canonically isomorphic to the full corner $p_X I_X p_X$ of the ideal I_X. Since E_X is a simple loop on $|\alpha|$ vertices, we have that $C^*(E_X) \cong C(\mathbb{T}, M_{|\alpha|}(\mathbb{C}))$ (see [34, Example 2.14] for details of this). Since E_X is a finite graph, Theorem 1.6(c)] implies that the gauge-invariant ideals of $C^*(E_X)$ are in one-to-one correspondence with certain subsets of E_X^0, and therefore the number of gauge-invariant ideals of $C^*(E_X)$ is finite. However, $C(\mathbb{T}, M_n(\mathbb{C}))$ has uncountably many ideals (corresponding to the closed subsets of $\mathbb{T}$) so we may conclude that $C^*(E_X)$ has uncountably many ideals that are not gauge invariant.

Because $C^*(E_X)$ is canonically isomorphic to $p_X I_X p_X$, it follows that $p_X I_X p_X$ has uncountably many ideals that are not gauge invariant. Furthermore, since $p_X I_X p_X$ is a full corner of I_X, the Rieffel Correspondence $I \mapsto p_X I p_X$ is an isomorphism from the lattice of ideals of I_X onto the lattice of ideals of $p_X I_X p_X$ (see [36, Theorem 3.22] and [36, Proposition 3.24]). In addition, since $\gamma_z(p_X a p_X) = p_X \gamma_z(a) p_X$ for all $a \in A$ and for all $z \in \mathbb{T}$, it follows that the isomorphism $I \mapsto p_X I p_X$ takes gauge-invariant ideals to gauge-invariant ideals. Because $p_X I_X p_X$ has uncountably many ideals that are not gauge invariant, it follows that I_X has uncountably many ideals that are not gauge invariant. But since any ideals of I_X are also ideals of $C^*(E)$ (recall that if I is an ideal off a C^*-algebra A, and if J is an ideal of I, then J is an ideal of A), we may conclude that $C^*(E)$ has an uncountable number of ideals that are not gauge invariant. □

Theorem 1.19. *A row-finite graph E satisfies Condition (K) if and only if all ideals of $C^*(E)$ are gauge invariant.*

Proof. Suppose E satisfies Condition (K). If I is an ideal in $C^*(E)$, then we may let $H = \{v \in E^0 : p_v \in I\}$ and proceed exactly as in the first paragraph of the proof of Theorem 1.6(a)] to form the map $q \circ \rho : C^*(E \backslash H) \to C^*(E)/I$, which is nonzero on the projections of the generating Cuntz-Krieger $(E \backslash H)$-family. By Proposition 1.17, the graph $E \setminus H$ satisfies Condition (L), and thus we may use the Cuntz-Krieger Uniqueness Theorem to conclude that

$q \circ \rho$ is injective. Hence the quotient map q is injective and $I = I_H$. By Lemma 1.11, the ideal I is gauge invariant.

Conversely, suppose that E does not satisfy Condition (K). By Proposition 1.17 there exists a saturated hereditary subset H such that the graph $E \setminus H$ does not satisfy Condition (L). It follows that $E \setminus H$ contains a loop without an exit, and therefore Lemma 1.18 implies that $C^*(E \setminus H)$ contains an ideal that is not gauge invariant. If γ denotes the gauge action on $C^*(E)$, then because I_H is gauge invariant, γ descends to a gauge action γ^{I_H} on $C^*(E)/I_H$. Furthermore, since $C^*(E \setminus H)$ is canonically isomorphic to $C^*(E)/I_H$ by Theorem 1.6(b), it follows that $C^*(E)/I_H$ contains an ideal J that is not gauge invariant with respect to the gauge action γ^{I_H}. Because the quotient map $q : C^*(E) \to C^*(E)/I_H$ has the property that $q \circ \gamma_z = \gamma_z^{I_H}$ for all $z \in \mathbb{T}$ (to see this simply verify the equality holds on generators) we see that $q^{-1}(J)$ is an ideal in $C^*(E)$ that is not gauge invariant. □

Corollary 1.20. *If E is a row-finite graph satisfying Condition (K), then all ideals of $C^*(E)$ are gauge invariant, and the map $H \mapsto I_H$ is a lattice isomorphism from the saturated hereditary subsets of E^0 onto the ideals of $C^*(E)$.*

Example 1.21. Since the graph E of Example 1.8 satisfies Condition (K), we see all of the ideals of $C^*(E)$ are gauge invariant and the lattice of ideals obtained in Example 1.8 describes all the ideals of $C^*(E)$. In particular, $C^*(E)$ has exactly three proper nontrivial ideals.

1.3. **Simplicity of Graph Algebras.** We shall now use our knowledge of gauge-invariant ideals to provide a characterization of simplicity for C^*-algebras of row-finite graphs. The amazing thing about this result is that it is a statement about all ideals — not simply the gauge-invariant ones.

Definition 1.22. We say that a graph E is *cofinal* if for every $v \in E^0$ and every infinite path $\alpha \in E^\infty$, there exists $i \in \mathbb{N}$ for which $v \geq s(\alpha_i)$.

In other words, E is cofinal if every vertex in E can reach every infinite path in E.

Theorem 1.23. *Let E be a row-finite graph. Then the following are equivalent.*

1. *$C^*(E)$ is simple*
2. *E satisfies Condition (L), E is cofinal, and if $v, w \in E^0$ with v a sink, then $w \geq v$*
3. *E satisfies Condition (K), E is cofinal, and if $v, w \in E^0$ with v a sink, then $w \geq v$*
4. *E satisfies Condition (L) and E^0 has no saturated hereditary subsets other than $\emptyset$ and E^0*
5. *E satisfies Condition (K) and E^0 has no saturated hereditary subsets other than $\emptyset$ and E^0*

Proof. (1) $\implies$ (2) Suppose that E is simple. Since $C^*(E)$ does not contain any ideals that are not gauge invariant, and by Lemma 1.18 E does not contain a loop with no exits. Hence E satisfies Condition (L).

Next let $\alpha \in E^\infty$ be an infinite path in E. Define

$$H := \{v \in E^0 : v \not\geq s(\alpha_i) \text{ for all } i \in \mathbb{N}\}.$$

It is straightforward to verify that H is saturated and hereditary. Because $C^*(E)$ is simple, the only gauge-invariant ideal of $C^*(E)$ are $\{0\}$ and $C^*(E)$, and it follows from Theorem 1.6(a) that the only saturated hereditary subsets of E^0 are $\emptyset$ and E^0. Since H is not equal to all of E^0 (the vertex $s(\alpha_1) \notin H$, for example), we must have that $H = \emptyset$. But then every vertex in E^0 can reach the infinite path α.

Finally, let $v \in E^0$ be a sink. If we let $H := \{w \in E^0 : w \not\geq v\}$, then one can verify that H is a saturated hereditary subset. As in the previous paragraph we must have that H equals either $\emptyset$ or E^0. Since $v \notin H$, we must have $H = \emptyset$. But then every vertex in E can reach v.

(2) $\implies$ (3) It suffices to show that under the hypotheses of (2), E satisfies Condition (K). Let v be the base point of a simple loop α. Since E satisfies Condition (L), it follows that α has an exit e, with $s(e) = s(\alpha_i)$ from some $i \in \{1, 2, \ldots, |\alpha|\}$. If we consider the infinite path $\alpha\alpha\alpha\ldots$, then because E is cofinal we know that $r(e)$ can reach this infinite path, and thus $r(e)$ can reach v. Let μ be the shortest path with $s(\mu) = r(e)$ and $r(\mu) = v$. Then $\alpha_1 \ldots \alpha_{i-1} e\mu$ is a simple loop based at v that is distinct from α. Hence there are two simple loops based at v, and since v was arbitrary, E satisfies Condition (K).

(3) $\implies$ (4) Since Condition (K) is a stronger condition than Condition (L), we have that E satisfies Condition (L). We shall suppose that H is a saturated hereditary subset with $H \neq \emptyset$ and $H \neq E^0$, and arrive at a contradiction. Choose $v \in E^0 \setminus H$. Since H is nonempty and hereditary, we know that there are vertices in H that cannot reach v. Thus, due to our hypotheses, v is not a sink. Since E is row-finite and since H is saturated, it must be the case that there is an edge $e_1 \in E^1$ with $s(e_1) = v$ and $r(e_1) \notin H$. Since $r(e_1) \notin H$ we may repeat this argument to produce and edge $e_2 \in E^1$ with $s(e_2) = r(e_1)$ and $r(e_2) \notin H$. Continuing in this fashion we produce an infinite path $e_1e_2e_3\ldots$ with the property that $r(e_i) \notin H$ for all $i \in \mathbb{N}$. But since H is nonempty and hereditary, there are vertices in H that cannot reach $r(e_i)$ for any $i \in \mathbb{N}$. This contradicts the fact that E is cofinal. Hence we may conclude that the only saturated hereditary subsets of E^0 are $\emptyset$ and E^0.

(4) $\implies$ (5) It suffices to show that under the hypotheses of (4), E satisfies Condition (K). Let v be the base point of a simple loop α. Since E satisfies Condition (L), it follows that α has an exit e, with $s(e) = s(\alpha_i)$ from some $i \in \{1, 2, \ldots, |\alpha|\}$. If we let

$$H = \{w \in E^0 : w \not\geq s(\alpha_i) \text{ for all } i = 1, 2, \ldots, |\alpha|\},$$

then one can verify that H is a saturated hereditary subset. By hypothesis, either $H = \emptyset$ or $H = E^0$. Since the vertex $v \notin H$, we must have $H = \emptyset$. But then every vertex in E can reach the vertices on the loop α, and hence every vertex can reach v. Let μ be the shortest path with $s(\mu) = r(e)$ and $r(\mu) = v$. Then $\alpha_1 \ldots \alpha_{i-1} e \mu$ is a simple loop based at v that is distinct from α. Hence there are two simple loops based at v, and since v was arbitrary, E satisfies Condition (K).

(5) $\Longrightarrow$ (1) If E satisfies Condition (K), then Theorem 1.19 implies that every ideal of $C^*(E)$ is gauge invariant. The result then follows from Theorem 1.6(a).

□

Corollary 1.24. *If E is a row-finite graph with two or more sinks, then $C^*(E)$ is not simple.*

Proof. If v_1 and v_2 are sinks in E, then v_1 cannot reach v_2. Thus the hypotheses of (2) in Theorem 1.23 are not satisfied. □

Corollary 1.25. *If E is a row-finite graph containing a sink, and if $C^*(E)$ is simple, then E contains no loops and no infinite paths.*

Proof. Let v be a sink in E. If α is a loop in E, then v cannot reach the infinite path $\alpha\alpha\alpha\ldots$, which implies that E is not cofinal and the hypotheses of (2) in Theorem 1.23 are not satisfied. Similarly, if α is an infinite path. □

As shown in the above corollary, simplicity of $C^*(E)$ imposes restrictions on the number of sinks and the presence of loops. In fact, more can be said about simple C^*-algebras of row-finite graphs: they are all either AF-algebras or purely infinite algebras.

Remark 1.26. A C^*-algebra is an AF-algebra (AF stands for *approximately finite-dimensional*) if it can be written as the closure of the increasing union of finite-dimensional C^*-algebras; or, equivalently, if it is the direct limit of a sequence of finite-dimensional C^*-algebras. It has been shown in [27, Theorem 2.4] that if E is a row-finite graph, then $C^*(E)$ is AF if and only if E has no loops.

Remark 1.27. If A is a C^*-algebra, we say that a C^*-subalgebra B of A is a *hereditary* subalgebra if $bab' \in B$ for all $a \in A$ and $b, b' \in B$ (or, equivalently, if $a \in A_+$ and $b \in B_+$ the inequality $a \leq b$ implies $a \in B$). Two projections p and q in a C^*-algebra A are said to be *equivalent* if there exists $u \in A$ such that $p = uu^*$ and $q = u^*u$, and a projection p is said to be *infinite* if it is equivalent to a proper subprojection.

A simple C^*-algebra A is *purely infinite* if every nonzero hereditary subalgebra of A contains an infinite projection. (The definition of purely infinite for non-simple C^*-algebra is more complicated, see [25].) It has been shown in [2, Proposition 5.3] and [27, Theorem 3.9] that if E is a row-finite graph, then every nonzero hereditary subalgebra of $C^*(E)$ contains an infinite projection if and only if E satisfies Condition (L) and every vertex in E connects to a loop. Combined with Theorem 1.23, this allows us to characterize purely infinite simple C^*-algebras of row-finite graphs.

In fact, we have the following dichotomy for simple C^*-algebras of row-finite graphs.

Proposition 1.28 (The Dichotomy for Simple Graph Algebras)**.** *Let E be a row-finite graph. If $C^*(E)$ is simple, then either*

(1) *$C^*(E)$ is an AF-algebra if E contains no loops; or*
(2) *$C^*(E)$ is purely infinite if E contains a loop.*

Proof. If E has no loops, the fact that $C^*(E)$ is an AF-algebra follows [27, Theorem 2.4]. On the other hand, if E contains a loop α, then since $C^*(E)$ is simple we know from Theorem 1.23(2) that E is cofinal, and every vertex in E can reach the infinite path $\alpha\alpha\alpha\ldots$. Thus every vertex in E can reach a loop. Furthermore, Theorem 1.23(2) also tells us that E satisfies Condition (L), and thus [2, Proposition 5.3] implies that $C^*(E)$ is purely infinite. $\square$

Remark 1.29. AF-algebras and purely infinite C^*-algebras are very different. An AF-algebra, being the direct limit of finite-dimensional C^*-algebras, is close to being a finite-dimensional C^*-algebra, and as a result cannot contain any infinite projections. On the other hand, purely infinite C^*-algebras contain an abundance of infinite projections — one in every nonzero hereditary subalgebra — which shows that they are very far from being finite dimensional C^*-algebras. As the dichotomy for simple graph algebras shows, the presence of loops in a graph E causes the associated C^*-algebra $C^*(E)$ to be spacious, in the sense that each loop results in the existence of infinite projections $C^*(E)$.

1.4. **Concluding Remarks.** With the results of this section was has a very good understanding of the gauge-invariant ideals in the C^*-algebra of a row-finite graph, as well as simplicity of $C^*(E)$. However, one may ask: What about general ideals? Can one describe the structure of *all* ideals of $C^*(E)$, even when E does not satisfy Condition (K)? This question has been answered affirmatively by Hong and Szymański in [17].

One way of describing the ideals in a C^*-algebra is in terms of primitive ideals. An ideal is *primitive* if it is the kernel of an irreducible representation (and, for separable C^*-algebras, an ideal is primitive if and only if it is prime). The set of primitive ideals in a C^*-algebra A is denoted by $\operatorname{Prim} A$, and every ideal in A is the intersection of the primitive ideals containing it [36, Proposition A.17]. Furthermore, for an ideal I of A, the set $h(I) := \{P \in \operatorname{Prim} A : I \subseteq P\}$ are the closed sets of a topology on $\operatorname{Prim} A$ [36, Proposition A.27], and $\operatorname{Prim} A$ endowed with this topology is called the *primitive ideal space of* A. Thus if one can describe the set $\operatorname{Prim} A$ as well as the topology on $\operatorname{Prim} A$, one has a description of all ideals in A.

In [17] Hong and Szymański have carried out this program for graph algebras. They give a description of the primitive ideals in $C^*(E)$ in terms of maximal tails of vertices, and they also give a description of the topology on the space $\operatorname{Prim} C^*(E)$. (In fact they do this for arbitrary graphs, without any assumption of row-finiteness!) As expected, this description is fairly involved (even if one restricts to the row-finite case), so we will not attempt to state it here.

2. C^*-algebras of Arbitrary Graphs

In the previous section we restricted our attention to row-finite graphs to avoid complications that arise when infinite emitters are present. This is fairly common in the subject, and when the theory of graph C^*-algebras was developed, theorems were often proven first in the row-finite case, and later extended to the general setting.

The theory of C^*-algebras of arbitrary graphs is significantly different from the theory of C^*-algebras of row-finite graphs. Although theorems for row-finite graph algebras sometimes remain true when one removes the word "row-finite" from their statements, it is not uncommon for new phenomena to appear in the non-row-finite case that require substantially new descriptions and theorems. More importantly, many of the proofs of theorems for row-finite graph algebras rely heavily on the non-row-finite assumption so that in the general setting entirely new methods and techniques must be developed to prove results.

In this section we will describe a construction called "desingularization" that allows one to bootstrap results from the row-finite case to the general setting. If E is an arbitrary graph, then one can "desingularize" E to form a row-finite graph F with no sinks that has the property that $C^*(E)$ is isomorphic to a full corner of $C^*(F)$. This allows one to use Morita Equivalence to study $C^*(E)$ in terms of $C^*(F)$.

In this section we will frequently draw graphs that have an infinite number of edges between vertices. We will use the notation

$$v \xrightarrow{(\infty)} w$$

in our graphs to indicate that there are a countably infinite number of edges from v to w.

In order to desingularize graphs, we will need to remove sinks and infinite emitters.

Definition 2.1. If E is a graph and v_0 is a sink in E, then by *adding a tail* at v_0 we mean attaching a graph of the form

$$v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots$$

to E at v_0.

Definition 2.2. If E is a graph and v_0 is an infinite emitter in E, then by *adding a tail at* v_0 we mean performing the following process: We first list the edges $g_1, g_2, g_3, \dots$ of $s^{-1}(v_0)$. Then we add a graph of the form

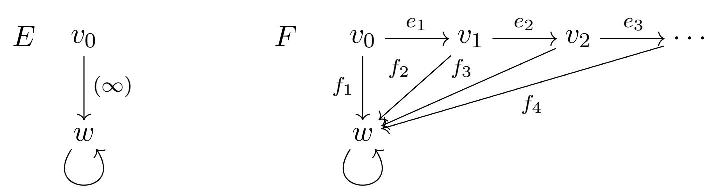
$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

to E at v_0, remove the edges in $s^{-1}(v_0)$, and for every $g_j \in s^{-1}(v_0)$ we draw an edge f_j from v_{j-1} to $r(g_j)$. We will find it convenient to use the following notation: For any $g_j \in s^{-1}(v_0)$ we let $\alpha_{v_0}^{g_j}$ denote the path $\alpha_{v_0}^{g_j} := e_1 e_2 \dots e_{j-1} f_j$ in F.

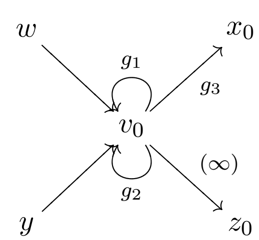
Definition 2.3. If E is a graph, then a *desingularization* of E is a graph F formed by adding a tail to every sink and infinite emitter of E.

Remark 2.4. We speak of "a" desingularization because the process of adding a tail to an infinite emitter is not unique; it depends on the ordering of the edges in $s^{-1}(v_0)$. Thus there may be different graphs F that are desingularizations of E. In addition, one can see that a desingularization of a graph is always row-finite and has no sinks.

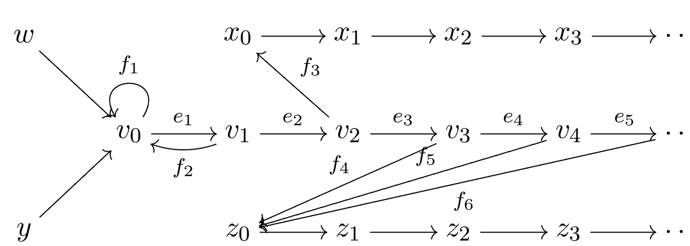
Example 2.5. Here is an example of a graph E and a desingularization F of E.



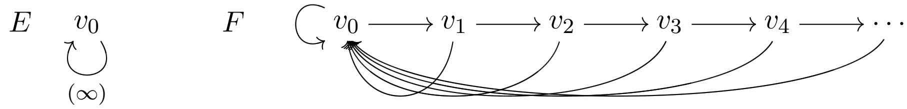
Example 2.6. Suppose E is the following graph:



Let us label the edges from v_0 to z_0 as $\{g_4, g_5, g_6, \dots\}$. Then a desingularization of E is given by the following graph F.



Example 2.7. If E is the $\mathcal{O}_\infty$ graph shown here, then a desingularization is given by the graph F:



The following fact is what will allow us to use desingularization to extend results for row-finite graph algebras to the general setting.

Theorem 2.8. *Let E be a graph. If F is a desingularization of E and p_{E^0} is the projection in $M(C^*(F))$ described in Lemma 1.13, then $C^*(E)$ is isomorphic to the corner $p_{E^0}C^*(F)p_{E^0}$, and this corner is full.*

Proof. Let $\{s_e, p_v : e \in F^1, v \in F^0\}$ be a Cuntz-Krieger F-family that generates $C^*(F)$.

For any $z \in \mathbb{T}$ we see that

$$\{s_e : e \in F^1 \text{ and } r(e) \notin E^0\} \cup \{zs_e : e \in F^1 \text{ and } r(e) \in E^0\} \cup \{p_v : v \in F^0\}$$

is a Cuntz-Krieger F-family, and thus induces a homomorphism $\beta_z : C^*(F) \to C^*(F)$ with $\beta_z(p_v) = p_v$ and

$$\beta_z(s_e) = \begin{cases} s_e & \text{if } r(e) \notin E^0 \\ zs_e & \text{if } r(e) \in E^0. \end{cases}$$

Furthermore, $\beta_{\overline{z}}$ is an inverse for β_z, so $\beta_z \in \operatorname{Aut} C^*(F)$, and we have defined a gauge action $\beta : \mathbb{T} \to \operatorname{Aut} C^*(F)$.

For $v \in E^0$ we define $q_v := p_v$, and for $e \in E^1$ we define

$$t_e := \begin{cases} s_{\alpha^e_{s(e)}} & \text{if } s(e) \text{ is an infinite emitter} \\ s_e & \text{if } s(e) \text{ is not an infinite emitter} \end{cases}$$

where $\alpha^e_{s(e)}$ is the path in F described in Definition 2.2. It is straightforward to verify that $\{t_e, q_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family (recall that if v_0 is an infinite emitter in E, then the third Cuntz-Krieger relation does not impose any requirements on q_{v_0}). Thus, by the universal property of $C^*(E)$, there exists a homomorphism $\rho : C^*(E) \to C^*(F)$ taking the generating partial isometries to the t_e's and the generating projections to the q_v's.

Let γ denote the standard gauge action on $C^*(E)$. Since the only edge of the path $\alpha^e_{s(e)} = e_1 \dots e_{j-1} f_j$ whose range is in E^0 is f_j we see that

$$\beta_z(s_{\alpha^e_{s(e)}}) = \beta_z(s_{e_1} \dots s_{e_{j-1}} s_{f_j}) = s_{e_1} \dots s_{e_{j-1}}(zs_{f_j}) = zs_{e_1} \dots s_{e_{j-1}} s_{f_j} = zs_{\alpha^e_{s(e)}}.$$

Thus $\beta_z \circ \rho$ and $\rho \circ \gamma_z$ agree on the generators of $C^*(E)$, and consequently $\beta_z \circ \rho = \rho \circ \gamma_z$. Since $q_v \neq 0$ for all $v \in E^0$, the Gauge-Invariant Uniqueness Theorem tells us that ρ is injective. Thus ρ is an isomorphism onto $\operatorname{im} \rho = C^*(\{t_e, q_v : e \in E^1, v \in E^0\})$.

We see that $p_{E^0}q_v = p_{E^0}p_v = p_v$ for all $v \in E^0$. Furthermore, when $e \in E^1$ with $s(e)$ not an infinite emitter we have $p_{E^0}t_e = p_{E^0}s_e = s_e = t_e$, and when $e \in E^1$ with $s(e)$ an infinite emitter we have $p_{E^0}t_e = p_{E^0}s_{\alpha^e_{s(e)}} = s_{\alpha^e_{s(e)}} = t_e$. Thus $\operatorname{im}\rho$ is contained the corner $p_{E^0}C^*(F)p_{E^0}$.

Conversely, since $a \mapsto p_{E^0}ap_{E^0}$ is continuous and linear

$$\begin{aligned} p_{E^0}C^*(F)p_{E^0} &= \overline{\operatorname{span}}\{p_{E^0}s_\alpha s_\beta^* p_{E^0} : \alpha, \beta \in F^*, r(\alpha) = r(\beta)\} \\ &= \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^*, r(\alpha) = r(\beta), s(\alpha) \in E^0, r(\alpha) \in E^0\}. \end{aligned}$$

Any path α in F whose source is in E^0 may be written as $\alpha_1 \dots \alpha_k e_1 e_2 \dots e_n$, where each α_i is either an edge in E^1 or a path of the form $\alpha_s(e)^e$ for $e \in E^1$, and where $e_1 \dots e_n$ is a path along a tail. Thus to show that $p_{E^0}C^*(F)p_{E^0}$ is contained in $\operatorname{im}\rho$ it suffices to show that $s_{e_1\dots e_n}s^*_{e_1\dots e_n}$ is contained in $\operatorname{im}\rho$. We shall show this by induction on n. If $n = 0$, then $s_{e_1\dots e_n}s^*_{e_1\dots e_n} = p_{s(e)} \in \operatorname{im}\rho$. Assume that $s_{e_1\dots e_n}s^*_{e_1\dots e_n} \in \operatorname{im}\rho$. Then there are two edges, e_{n+1} and f_j, whose source is $r(e_n)$. Hence $p_{r(e_n)} = s_{e_{n+1}}s^*_{e_{n+1}} + s_{f_j}s^*_{f_j}$, and

$$s_{e_1\dots e_{n+1}}s^*_{e_1\dots e_{n+1}} = s_{e_1\dots e_n}(p_{r(e_n)} - s_{f_j}s^*_{f_j})s^*_{e_1\dots e_n} = s_{e_1\dots e_n}s^*_{e_1\dots e_n} - s_{\alpha^e_{s(e)}}s^*_{\alpha^e_{s(e)}}$$

which is in $\operatorname{im}\rho$. Thus $p_{E^0}C^*(F)p_{E^0} = \operatorname{im}\rho$.

Finally, to see that $p_{E^0}C^*(F)p_{E^0}$ is full, suppose I is an ideal containing this corner. Then $p_v \in p_{E^0}C^*(F)p_{E^0} \subseteq I$ when $v \in E^0$. When $v \in F^0 \setminus E^0$, then $v = r(e_1 \dots e_n)$ for some path $e_1 \dots e_n$ on an added tail. Thus $s_{e_1\dots e_n} = p_{s(e_1)}s_{e_1\dots e_n} \in I$, and $p_{r(e)} = s^*_{e_1\dots e_n}s_{e_1\dots e_n} \in I$. Since $\{p_v : v \in F^0\} \subseteq I$, and F is row-finite, it follows from Theorem 1.6(a) that I is all of $C^*(F)$. □

The advantage of the process of desingularization is that it is very concrete, and it allows us to use the row-finite graph F to see how the properties of $C^*(E)$ are reflected in the graph E. We will see examples of this in the following proofs, as we show how to extend results for C^*-algebras of row-finite graphs to general graph algebras.

Theorem 2.9. *Let E be a graph. The graph algebra $C^*(E)$ is an AF-algebra if and only if E has no loops.*

Proof. Let F be a desingularization of E. Since F is row-finite, it follows from [27, Theorem 2.4] that $C^*(F)$ is an AF-algebra if and only if F has no loops. It follows from [9, Theorem 9.4] that Morita equivalence preserves AF-ness for separable C^*-algebra. Thus $C^*(E)$ is an AF-algebra if and only if F has no loops. Since E has no loops if and only if F has no loops, the result follows. □

Theorem 2.10. *Let E be a graph. If E satisfies Condition (L) and every vertex in E connects to a loop in E, then there exists an infinite projection in every nonzero hereditary subalgebra of $C^*(E)$.*

Proof. Let F be a desingularization of E. We see that if E satisfies Condition (L), then F satisfies Condition (L). Also, if every vertex in E connects to a loop in E, then every vertex in F connects to a loop in F. It then follows from [2, Proposition 5.3] that there exists an infinite projection in every nonzero hereditary subalgebra of $C^*(F)$. Since this is a property that is preserved by passing to corners, there exists an infinite projection in every nonzero hereditary subalgebra of $C^*(E)$. $\square$

Remark 2.11. The corollary of Theorem 2.10 is also true; the proof of [27, Theorem 3.9] works for arbitrary graphs.

In each of the above theorems we have seen that we have the same descriptions as in the row-finite case, and basically each of the theorems for row-finite graph algebras remains true when we remove the term "row-finite" from the theorem's statement. Next we characterize simplicity for C^*-algebras of arbitrary graphs. In this situation we shall see that there are new phenomena occurring, which will require a description different from that in the row-finite case.

The following theorem generalizes the characterization given in Theorem 1.23(2).

Theorem 2.12. *If E is a graph, then $C^*(E)$ is simple if and only if E has the following four properties:*

(1) *E satisfies Condition (L),*
(2) *E is cofinal,*
(3) *if $v, w \in E^0$ with v a sink, then $w \geq v$, and*
(4) *if $v, w \in E^0$ with v an infinite emitter, then $w \geq v$.*

Proof. Let F be a desingularization of E. Since simplicity is preserved by Morita equivalence, $C^*(E)$ is simple if and only if $C^*(F)$ is simple. But since F is row-finite with no sinks, Theorem 1.23(2) implies that $C^*(F)$ is simple if and only if F satisfies Condition (L) and F is cofinal. We see that F satisfies Condition (L) if and only if E satisfies Condition (L), giving (1). Also, we see that the infinite paths of F are of two types: either they come from infinite paths in E or they are paths that go along the tails added in forming the desingularization. Thus F is cofinal if and only if every vertex w in E can reach every infinite path in E, which occurs if and only if every vertex w in F can reach every infinite path in E, every sink in E, and every infinite emitter in E; this gives (2), (3), and (4). $\square$

Definition 2.13. We say that a graph E is *transitive* if for every $v, w \in E^0$ it is the case that $v \geq w$ and $w \geq v$.

Corollary 2.14. *If E is a graph in which every vertex is an infinite emitter, then $C^*(E)$ is simple if and only if E is transitive.*

Using our characterization of simplicity, we can now show that the dichotomy for simple graph algebras holds even when the graph is not row-finite.

Proposition 2.15 (The Dichotomy for Simple Graph Algebras)**.** *Let E be a graph. If $C^*(E)$ is simple, then either*

(1) *$C^*(E)$ is an AF-algebra if E contains no loops; or*
(2) *$C^*(E)$ is purely infinite if E contains a loop.*

Proof. If E has no loops, the fact that $C^*(E)$ is an AF-algebra follows Theorem 2.9. On the other hand, if E contains a loop α, then since $C^*(E)$ is simple we know from Theorem 2.12 that E is cofinal, and every vertex in E can reach the infinite path $\alpha\alpha\alpha\ldots$. Thus every vertex in E can reach a loop. Furthermore, Theorem 2.12 also tells us that E satisfies Condition (L), and thus Theorem 2.10 implies that $C^*(E)$ is purely infinite. $\square$

Finally, we shall use the process of desingularization to analyze the ideal structure of C^*-algebras corresponding to graphs satisfying Condition (K). This will be more involved than our prior applications of desingularization, and we shall see that that structure of ideals in $C^*(E)$ will require more than just the saturated hereditary subsets of E as it does in the row-finite case.

We first need to identify the saturated hereditary subsets of F in terms of E. Recall that if E is a directed graph, then a set $H \subseteq E^0$ is *hereditary* if whenever $e \in E^1$ with $s(e) \in H$, then $r(e) \in H$. A hereditary set H is called *saturated* if every vertex that is not a sink or infinite emitter and that feeds only into H is itself in H; that is, if

$$v \text{ not a sink or infinite emitter, and } \{r(e) \mid s(e) = v\} \subseteq H \text{ implies } v \in H.$$

Let E be a graph that satisfies Condition (K). When E is row-finite Theorem 1.6(a) and Theorem 1.19 show that the saturated hereditary subsets of E correspond to the ideals of $C^*(E)$ via the map $H \mapsto I_H$, where I_H is the ideal generated by $\{p_v : v \in H\}$. When E is not row-finite, this is not the case. For an arbitrary graph E, one can check that $H \mapsto I_H$ is still injective, just as shown in the proof of Theorem 1.6(a). However, it is no longer true that this map is surjective; that is, there may exist ideals in $C^*(E)$ that are not of the form I_H for some saturated hereditary set H. The reason the proof for row-finite graphs no longer works is that if I is an ideal, then $\{s_e + I, p_v + I\}$ will not necessarily be a Cuntz-Krieger $E \setminus H$-family for the graph $E \setminus H$ defined in Theorem 1.6(a). (And, consequently, it is sometimes not true that $C^*(E)/I_H \cong C^*(E \setminus H)$.) To describe an ideal in $C^*(E)$ we will need a saturated hereditary subset and one other piece of information. Loosely speaking, this additional piece of information tells us how close $\{s_e + I, p_v + I\}$ is to being a Cuntz-Krieger $E \setminus H$-family.

Definition 2.16. Given a saturated hereditary subset $H \subseteq E^0$, we define the *breaking vertices* of H to be the set

$$B_H := \{v \in E^0 : v \text{ is an infinite-emitter and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

We see that B_H is the set of infinite-emitters that point to a finite (and nonzero) number of vertices not in H. Also, since H is hereditary, B_H is disjoint from H.

Now fix a saturated hereditary subset H of E and let S be any subset of B_H. Let $\{s_e, p_v\}$ be the canonical generating Cuntz-Krieger E-family and define

$$I_{(H,S)} := \text{the ideal in } C^*(E) \text{ generated by } \{p_v : v \in H\} \cup \{p_{v_0}^H : v_0 \in S\},$$

where $p_{v_0}^H$ is the *gap projection* defined by

$$p_{v_0}^H := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e) \notin H}} s_e s_e^*.$$

Note that the definition of B_H ensures that the sum on the right is finite.

Definition 2.17. Let E be a graph. We say that (H,S) is an *admissible pair* for E if H is a saturated hereditary subset of vertices of E and $S \subseteq B_H$. For a fixed graph E we order the collection of admissible pairs for E by defining $(H,S) \leq (H',S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$.

Example 2.18. Let E be the graph

$$\begin{tikzcd} v \arrow[r, "(\infty)"] & w \arrow[r, "(\infty)"] \arrow[d] & x \\ & y & \end{tikzcd}$$

Then the saturated hereditary subsets of E are E^0, $\{w,x,y\}$, $\{x,y\}$, $\{x\}$, $\{y\}$, and $\emptyset$. Also $B_{\{x\}} = \{w\}$, and $B_H = \emptyset$ for all other saturated hereditary H in E. Thus the admissible pairs of E are:

$$(E^0,\emptyset), (\{w,x,y\},\emptyset), (\{x,y\},\emptyset), (\{x\},\{w\}), (\{x\},\emptyset), (\{y\},\emptyset), (\emptyset,\emptyset)$$

and these admissible pairs are ordered in the following way.

$$\begin{tikzcd} & (E^0,\emptyset) \arrow[d, dash] & \\ & (\{w,x,y\},\emptyset) \arrow[dl, dash] \arrow[dr, dash] & \\ (\{x,y\},\emptyset) \arrow[d, dash] \arrow[drr, dash] & & (\{x\},\{w\}) \arrow[d, dash] \\ (\{y\},\emptyset) \arrow[dr, dash] & & (\{x\},\emptyset) \arrow[dl, dash] \\ & (\emptyset,\emptyset) & \end{tikzcd}$$

We shall show that the correspondence $(H,S) \mapsto I_{(H,S)}$ is an inclusion-preserving bijection. To do this we will first describe a correspondence between admissible pairs in E and saturated hereditary subsets of vertices in a desingularization of E.

Definition 2.19. Suppose that E is a graph and let F be a desingularization of E. Also let (H,S) be an admissible pair for E. We define a saturated hereditary subset $H_S \subseteq F^0$ as follows. We first define $\tilde{H} := H \cup \{v_n \in F^0 : v_n \text{ is on a tail added to a vertex in } H\}$. Now for each $v_0 \in S$ let N_{v_0} be the smallest nonnegative integer such that $r(f_j) \in H$ for all $j > N_{v_0}$. (The number N_{v_0} exists since $v_0 \in B_H$ implies that there must be a vertex on the tail added to v_0 beyond which each subsequent vertex points only to the next vertex on the tail and into H.) Define $T_{v_0} := \{v_n : v_n \text{ is on the tail added to } v_0 \text{ and } n \geq N_{v_0}\}$ and define

$$H_S := \tilde{H} \cup \bigcup_{v_0 \in S} T_{v_0}.$$

Note that for $v_0 \in B_H$ we have $v_0 \notin H_S$. Furthermore, the tail attached to v_0 will eventually be inside H_S if and only if $v_0 \in S$. It is easy to check that H_S is hereditary, and choosing N_{v_0} to be minimal ensures that H_S is saturated.

Example 2.20. Let E be the graph shown in Example 2.6. If we let $H = \{z_0\}$, then H is saturated hereditary and $B_H = \{v_0\}$. Suppose $S = \{v_0\}$. Then $\tilde{H} = \{z_0, z_1, z_2, \ldots\}$ and $N_{v_0} = 3$, so $T_{v_0} = \{v_3, v_4, v_5, \ldots\}$, and $H_S = \{z_0, z_1, \ldots, v_3, v_4, \ldots\}$.

In a similar manner we can see that $H_\emptyset = \{z_0, z_1, z_2, \ldots\}$.

Lemma 2.21. *Let E be a graph and let F be a desingularization of E. The map $(H,S) \mapsto H_S$ is an order-preserving bijection from the lattice of admissible pairs of E onto the lattice of saturated hereditary subsets of F.*

Proof. Let K be a saturated hereditary subset of F. Define

$$S_K := \{v_0 \in B_{K \cap E^0} : \text{ past a certain point all vertices on the tail added to } v_0 \text{ are in the set } K\}.$$

One can easily check that the map $K \mapsto (K \cap E^0, S_K)$ is an inverse for the map $(H,S) \mapsto H_S$, and that the map $(H,S) \mapsto H_S$ is inclusion preserving. □

To analyze the ideals of $C^*(E)$ we will make use of the Rieffel correspondence. Whenever two C^*-algebra A and B are Morita equivalent, there is a lattice isomorphism between the lattice of ideals of A and the lattice of ideals of B. When one of these C^*-algebras is a full corner of the other, this correspondence takes the following form:

Lemma 2.22. *Suppose A is a C^*-algebra, p is a projection in the multiplier algebra $M(A)$, and pAp is a full corner of A. Then the map $I \mapsto pIp$ is* *an order-preserving bijection from the ideals of A to the ideals of pAp; its inverse takes an ideal J in pAp to*

$$\overline{AJA} := \overline{\operatorname{span}}\{aba' : a, a' \in A \text{ and } b \in J\}.$$

Proof. Suppose I is an ideal in A. The continuity of $a \mapsto pap$ shows that pIp is closed in pAp, and $(pAp)(pIp)(pAp) = p(Ap)I(pA)p \subseteq pIp$ shows that pIp is an ideal in pAp. Furthermore,

$$\overline{A(pIp)A} = \overline{Ap(AIA)pA} = \overline{\overline{ApA}I\overline{ApA}} = \overline{AIA} = I.$$

Conversely, if J is an ideal in pAp, then

$$p\overline{AJA}p = \overline{pAJAp} = \overline{pA(pAp)J(pAp)Ap} = \overline{(pAp)J(pAp)} = J.$$

The above two paragraphs show that the maps under discussion are inverses of each other. It is also clear that these maps preserve ordering by inclusion. □

Proposition 2.23. *Let E be a graph and let F be a desingularization of E. Let p_{E^0} be the projection in $M(C^*(F)$ described in Lemma 1.13, and identify $C^*(E)$ with $p_{E^0}C^*(F)p_{E^0}$ as described in Theorem 2.8. If H is a saturated hereditary subset of E^0 and $S \subseteq B_H$, then then $p_{E^0}I_{H_S}p_{E^0} = I_{(H,S)}$.*

Proof. Let $\{s_e, p_v : e \in F^1, v \in F^0\}$ be a generating Cuntz-Krieger F-family. As shown in the proof of Theorem 2.8, the set $\{t_e, q_v : e \in E^1, v \in E^0\}$, where $q_v := p_v$ and

$$t_e := \begin{cases} s_{\alpha^e_{s(e)}} & \text{if } s(e) \text{ is an infinite emitter} \\ s_e & \text{if } s(e) \text{ is not an infinite emitter} \end{cases}$$

is a Cuntz-Krieger E-family that generates a C^*-subalgebra of $C^*(F)$ isomorphic to $C^*(E)$, and furthermore, this C^*-subalgebra is equal to the corner of $C^*(F)$ determined by p_{E^0}.

It follows from Lemma 1.11 that

$$I_{H_S} = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta)\}.$$

Thus

$$\begin{aligned} &p_{E^0}I_{H_S}p_{E^0} \\ &= \overline{\operatorname{span}}\{p_{E^0}s_\alpha s_\beta^* p_{E^0} : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta) \in H_S\} \\ &= \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^*, s(\alpha) \in E^0, s(\alpha) \in E^0, \text{ and } r(\alpha) = r(\beta) \in H_S\} \end{aligned}$$

If $s_\alpha s_\beta^*$ is an element of this ideal with $r(\alpha) = r(\beta) \in H$, then $s_\alpha s_\beta^*$ is of the form $t_\mu t_\nu^*$ for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu) \in H$, and hence $s_\alpha s_\beta^* \in I_{(H,S)}$. On the other hand, if $s_\alpha s_\beta^*$ is an element of this ideal with $r(\alpha) = r(\beta) \in T_{v_0}$ for some $v_0 \in S$, then $s_\alpha s_\beta^*$ is of the form $t_\mu s_{e_1 \ldots e_k} s_{e_1 \ldots e_k}^* t_\nu^*$ for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu) = v_0$. But then

$$\begin{aligned} s_\alpha s_\beta^* &= t_\mu s_{e_1 \ldots e_k} s_{e_1 \ldots e_k}^* t_\nu^* \\ &= t_\mu s_{e_1 \ldots e_{k-1}} (p_{v_{k-1}} - s_{f_k} s_{f_k}^*) s_{e_1 \ldots e_{k-1}}^* t_\nu^* \end{aligned}$$

$$
\begin{aligned}
&= t_\mu s_{e_1\ldots e_{k-1}} s^*_{e_1\ldots e_{k-1}} t^*_\nu - t_\mu s_{e_1\ldots e_{k-1} f_k} s^*_{e_1\ldots e_{k-1} f_k} t^*_\nu \\
&\quad \vdots \\
&= t_\mu t^*_\nu - \sum_{j=1}^{k} t_\mu t_{g_j} t^*_{g_j} t^*_\nu \\
&= t_\mu (q_{v_0} - \sum_{\substack{s(g)=v_0 \\ r(g)\notin H}} t_g t^*_g) t^*_\nu - \sum_{\substack{r(g_j)\in H \\ j\leq k}} t_{\mu g_j} t^*_{\mu g_j} \\
&= t_\mu q^H_{v_0} t^*_\nu - \sum_{\substack{r(g_j)\in H \\ j\leq k}} t_{\mu g_j} t^*_{\mu g_j} \\
&\in I_{(H,S)}
\end{aligned}
$$

Hence we have shown that $p_{E^0} I_{H_S} p_{E^0} \subseteq I_{(H,S)}$.

To verify the reverse inclusion we shall show that the generators $\{q_v : v \in H\} \cup \{q^H_{v_0} : v_0 \in S\}$ of $I_{(H,S)}$ are in $p_{E^0} I_{H_S} p_{E^0}$. Clearly for $v \in H$ we have $q_v = p_v = p_{E^0} p_v p_{E^0} \in p_{E^0} I_{H_S} p_{E^0}$, so all that remains to show is that for every $v_0 \in S$ we have $q^H_{v_0} \in p_{E^0} I_{H_S} p_{E^0}$.

Let $v_0 \in S$ and $n := N_{v_0}$. Then

$$
\begin{aligned}
p_{v_0} &= s_{e_1} s^*_{e_1} + s_{f_1} s^*_{f_1} \\
&= s_{e_1} p_{v_1} s^*_{e_1} + s_{f_1} s^*_{f_1} \\
&= s_{e_1} (s_{e_2} s^*_{e_2} + s_{f_2} s^*_{f_2}) s^*_{e_1} + s_{f_1} s^*_{f_1} \\
&= s_{e_1 e_2} p_{v_2} s^*_{e_1 e_2} + s_{e_1 f_2} s^*_{e_1 f_2} + s_{f_1} s^*_{f_1} \\
&\quad \vdots \\
&= s_{e_1\ldots e_n} s^*_{e_1\ldots e_n} + \sum_{j=1}^{n} t_{g_j} t^*_{g_j}
\end{aligned}
$$

Now since $r(e_n) = v_n \in H_S$ we see that $p_{v_n} \in I_{H_S}$ and hence $s_{e_n} = s_{e_n} p_{v_n} \in I_{H_S}$. Consequently, $s_{e_1\ldots e_n} s^*_{e_1\ldots e_n} \in I_{H_S}$. Similarly, whenever $r(g_j) \in H$, then $t_{g_j} t^*_{g_j} \in I_{H_S}$. Now, by definition, every g_j with $r(g_j) \notin H$ has $j < n$.

Therefore the above equation shows us that

$$
\begin{aligned}
q^H_{v_0} &= q_{v_0} - \sum_{\substack{s(g_j)=v_0 \\ r(g_j)\notin H}} t_{g_j} t^*_{g_j} \\
&= p_{v_0} - \sum_{\substack{s(g_j)=v_0 \\ r(g_j)\notin H}} t_{g_j} t^*_{g_j} \\
&= \sum_{\substack{r(g_j)\in H \\ j<n}} t_{g_j} t^*_{g_j} + s_{e_1\ldots e_n} s^*_{e_1\ldots e_n}
\end{aligned}
$$

which is an element of I_{H_S} by the previous paragraph. Hence $I_{H_S} \subseteq I_{H,S}$. ☐

Theorem 2.24. *Let E be a graph that satisfies Condition (K). Then the map $(H,S) \mapsto I_{(H,S)}$ is an inclusion-preserving bijection from admissible pairs for E onto the ideals of $C^*(E)$.*

Proof. Let F be a desingularization of E. First, it follows from Lemma 2.21 that the map $(H,S) \mapsto H_S$ is an order-preserving bijection from the admissible pairs of E onto the saturated hereditary subsets of F. Second, since the loops in E are in one-to-one correspondence with the loops in F, we see that F satisfies Condition (K); because F is row-finite, it follows from Theorem 1.6 and Theorem 1.19 that the map $H \mapsto I_H$ is an order-preserving bijection from the saturated hereditary subsets of F onto the ideal of $C^*(F)$. Third, we see from Theorem 2.8 and Lemma 2.22 that the map $I \mapsto p_{E^0} I p_{E^0}$ is an order-preserving bijection from the ideals of $C^*(F)$ onto the ideals of $C^*(E)$.

Composing these three maps gives $(H,S) \mapsto p_{E^0} I_{H_S} p_{E^0}$, and the result then follows from Proposition 2.23. ☐

Remark 2.25. When E does not satisfy Condition (K), the ideals $I_{(H,S)}$ are precisely the gauge-invariant ideals in $C^*(E)$ [1, Theorem 3.6]. In addition, although we have spoken of the collection of admissible pairs as being an ordered set, it is also a lattice and the map $(H,S) \mapsto I_{(H,S)}$ is a lattice isomorphism. This lattice structure is described in [7, §3], but because it is somewhat complicated we left it out of our discussion to avoid non-insightful technicalities.

Furthermore, we have already discussed how the quotient $C^*(E)/I_{(H,S)}$ is not necessarily isomorphic to $C^*(E \setminus H)$ because the collection $\{s_e + I_{(H,S)}, p_v + I_{(H,S)}\}$ may fail to satisfy the third Cuntz-Krieger relation at breaking vertices for H. However, one can show that $C^*(E)/I_{(H,S)}$ is isomorphic to $C^*(F_{H,S})$, where $F_{H,S}$ is the graph defined by

$$F^0_{H,S} := (E^0 \backslash H) \cup \{v' : v \in B_H \backslash S\}$$
$$F^1_{H,S} := \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \backslash S\}$$

and r and s are extended by $s(e') = s(e)$ and $r(e') = r(e)'$. (One can see that $F_{H,S}$ is formed by outsplitting $E \setminus F$ at the the breaking vertices not in S, and this adds a sink to $E \setminus H$ for each vertex in $B_H \setminus S$.) A construction of this isomorphism can be found in [1, Corollary 3.5]. One can also see that when $S = B_H$, we have $F_{H,S} = E \setminus H$. So if H is a saturated hereditary subset of E, then $C^*(E)/I_{(H,B_H)} \cong C^*(E \setminus H)$.

We conclude this section by mentioning that desingularization has also been used to generalize many other results for row-finite graph algebras to general graph algebras: In [7, Corollary 2.12] desingularization was used to extend the Cuntz-Krieger Uniqueness Theorem, in [7, §4] desingularization was used to extend the description of $\operatorname{Prim} C^*(E)$ when E satisfies Condition (K), in [8] desingularization was used to extend the computations of K-theory and Ext for $C^*(E)$, and in [11] desingularization was used to extend characterizations of liminal and Type I graph algebras to the general setting.

As with any construction, it is good to understand not only the uses of desingularization, but also its limitations. If we look at the proof of Theorem 2.8 we see that $C^*(E)$ is isomorphic to a full corner of $C^*(F)$. However, this isomorphism is not equivariant for the gauge actions on $C^*(E)$ and $C^*(F)$ — in fact, in order to apply the Gauge-Invariant Uniqueness Theorem in the proof of Theorem 2.8, we had to create a new gauge action β on $C^*(F)$. One of the consequences of this fact is that there is no obvious way to use desingularization to extend the Gauge-Invariant Uniqueness Theorem for row-finite graph algebras to general graph algebras. Currently, all known proofs of the Gauge-Invariant Uniqueness Theorem for arbitrary graphs either prove the result directly or use approximations by subalgebras that are isomorphic to C^*-algebras of finite graphs (see [35, §1] and [1, Theorem 2.1]. However, if it is possible, it might be interesting to have a proof of the Gauge-Invariant Uniqueness Theorem that uses desingularization.

3. K-theory of Graph Algebras

In K-theory one associates to each C^*-algebra A two abelian groups $K_0(A)$ and $K_1(A)$. These groups reflect a great deal of the structure of A, and they have a number of remarkable properties. Unfortunately, the subject of K-theory can be rather technical (in fact entire books [47, 37] have been written with the goal of giving the reader a mere *introduction* to the subject). Therefore in this section we will give a brief description of K-theory, survey the K-theory computations that have been accomplished for graph algebras, and discuss how classification theorems for C^*-algebras can be applied to graph algebras.

Definition 3.1. If A is a unital C^*-algebra, the group $K_0(A)$ is formed as follows: For each natural number n we let $\operatorname{Proj} M_n(A)$ be the set of projections in $M_n(A)$. By identifying $p \in \operatorname{Proj} M_n(A)$ with the projection $p \oplus 0$ in $\operatorname{Proj} M_{n+1}(A)$ formed by adding a row and column of zeros to the bottom and right of p, we may view $\operatorname{Proj} M_n(A)$ as a subset of $\operatorname{Proj} M_{n+1}(A)$. With this identification we let $\operatorname{Proj}_\infty(A) = \bigcup_{n=1}^\infty \operatorname{Proj} M_n(A)$. We define an equivalence relation on $\operatorname{Proj}_\infty(A)$ by saying $p \sim q$ if there exists $u \in \operatorname{Proj}_\infty(A)$ with $p = uu^*$ and $q = u^*u$. We let $[p]_0$ denote the equivalence class of $p \in \operatorname{Proj}_\infty(A)$. We define an addition on these equivalence classes by setting $[p]_0 + [q]_0$ equal to $\left[\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)\right]_0$. With this operation of addition, the equivalence classes of $\operatorname{Proj}_\infty(A)$ are an abelian semigroup. We define $K_0(A)$ to be the *Grothendieck group* of this semigroup; that is, $K_0(A)$ is the abelian group of formal differences

$$K_0(A) := \{[p]_0 - [q]_0 : p, q \in \mathrm{Proj}_\infty(A)\}$$

with $([p]_0 - [q]_0) + ([p']_0 - [q']_0) := ([p]_0 + [p']_0) - ([q]_0 + [q']_0)$.

Definition 3.2. The group $K_1(A)$ is defined using the groups $U(M_n(A))$ of unitary elements in $M_n(A)$. We embed $U(M_n(A))$ into $U(M_{n+1}(A))$ by $u \mapsto u \oplus 1$, where $u \oplus 1$ is the matrix formed by adding a 1 to the bottom right-hand corner and zeroes elsewhere in the right column and bottom row. We then let $U_\infty(A) := \bigcup_{n=1}^\infty U(M_n(A))$. We define an equivalence relation on $U_\infty(A)$ as follows: If $u \in U_m(A)$ and $v \in U_n(A)$, we write $u \sim v$ if there exists a natural number $k \geq \max\{m, n\}$ such that $\left(\begin{smallmatrix} u & 0 \\ 0 & 1_{k-n} \end{smallmatrix}\right)$ is homotopic to $\left(\begin{smallmatrix} v & 0 \\ 0 & 1_{k-m} \end{smallmatrix}\right)$ in $U_k(A)$ (i.e., there exists a continuous map $h : [0,1] \to U_k(A)$ such that $h(0) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1_{k-n} \end{smallmatrix}\right)$ and $h(1) = \left(\begin{smallmatrix} v & 0 \\ 0 & 1_{k-m} \end{smallmatrix}\right)$. We denote the equivalence class of $u \in U_\infty(A)$ by $[u]_1$. We define $K_1(A)$ to be

$$K_1(A) := \{[u]_1 : u \in U_\infty(A)\}$$

with addition given by $[u]_1 + [v]_1 := [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]_1$. It is true (but not obvious) that $K_1(A)$ is an abelian group.

The K-groups $K_0(A)$ and $K_1(A)$ can also be defined when A is nonunital. We refer the reader to [47] and [37] for these definitions as well as for details of the definitions in the unital case.

Remark 3.3. If $\phi : A \to B$ is a homomorphism between C^*-algebras, then ϕ induces homomorphisms $\phi_n : M_n(A) \to M_n(B)$ by $\phi((a_{ij})) = (\phi(a_{ij}))$. Since the ϕ_n's map projections to projections and unitaries to unitaries, they induce homomorphisms $K_0(\phi) : K_0(A) \to K_0(B)$ and $K_1(\phi) : K_1(A) \to K_1(B)$. This process is *functorial*: the identity homomorphism induces the identity map on K-groups, and $K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi)$ for $i = 1, 2$. Thus K_0 and K_1 are functors from the category of C^*-algebras to the category of abelian groups.

Remark 3.4. An *ordered abelian group* (G, G_+) is an abelian group G together with a distinguished subset $G_+ \subseteq G$ satisfying

$$(i)\ G_+ + G_+ \subseteq G_+, \quad (ii)\ G_+ \cap (-G_+) = \{0\}, \quad (iii)\ G_+ - G_+ = G.$$

We call G_+ the *positive cone* of G, and it allows us to define an ordering on G by setting $g_1 \leq g_2$ if and only if $g_2 - g_1 \in G_+$.

If A is a C^*-algebra, and we set

$$K_0(A)_+ := \{[p]_0 : p \in \mathrm{Proj}_\infty(A)\},$$

then $(K_0(A), K_0(A)_+)$ satisfies condition (i) above, but will not necessarily satisfy (ii) and (iii). However, if A is an AF-algebra, then $(K_0(A), K_0(A)_+)$ does satisfy conditions (ii) and (iii) and $(K_0(A), K_0(A)_+)$ is an ordered abelian group. (More generally, if A has an approximate unit consisting of projections, then $(K_0(A), K_0(A)_+)$ satisfies condition (iii), and if$A$ is also stably finite then $(K_0(A), K_0(A)_+)$ satisfies condition (ii).)

We shall often have to consider isomorphisms from the K-groups of a C^*-algebra to abelian groups, and frequently we will want these isomorphisms to preserve the ordering or take an element in the K-group to a distinguished element in the group. Therefore we establish the following notation.

Definition 3.5. Let A be a C^*-algebra, and let G be an abelian group and $g \in G$. If $p \in A$ is a projection, then we write $(K_0(A), [p]_0) \cong (G, g)$ if there is an isomorphism $\alpha : K_0(A) \to G$ with $\alpha([p]_0) = g$. If $G_+ \subseteq G$ is a positive cone of G, then we write $(K_0(A), K_0(A)_+) \cong (G, G_+)$ if there is an isomorphism $\alpha : K_0(A) \to G$ with $\alpha(K_0(A)_+) = G_+$, and we write $(K_0(A), K_0(A)_+, [p]_0) \cong (G, G_+, g)$ if there is an isomorphism $\alpha : K_0(A) \to G$ with $\alpha(K_0(A)_+) = G_+$ and $\alpha([p]_0) = g$.

Remark 3.6. If E is a graph and $v \in E^0$ is a vertex that is neither a sink nor an infinite emitter, then $p_v = \sum_{s(e)=v} s_e s_e^*$, and in $K_0(C^*(E))$ we have

$$[p_v]_0 = \left[\sum_{s(e)=v} s_e s_e^* \right]_0 = \sum_{s(e)=v} [s_e s_e^*]_0 = \sum_{s(e)=v} [s_e^* s_e]_0 = \sum_{s(e)=v} [p_{r(e)}]_0.$$

Theorem 3.9 says, among other things, that $K_0(C^*(E))$ is generated by the collection $\{[p_v]_0 : v \in E^0\}$ and that this collection is subject only to the above relations.

3.1. **Computing K-theory.** For our K-theory computation we will associate a matrix to a directed graph that will summarize the relations in Remark 3.6.

Definition 3.7. Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sinks. The *vertex matrix* of E is the (possibly infinite) $E^0 \times E^0$ matrix A_E whose entries are the non-zero integers

$$A_E(v, w) := \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}.$$

Remark 3.8. Let E be a row-finite graph and let $\bigoplus_{E^0} \mathbb{Z}$ denote the direct sum of copies of $\mathbb{Z}$ indexed by E^0 (i.e. sequences of integers that only have a finite number of nonzero terms). If E is row-finite, then each row of the matrix A_E contains a finite number of nonzero entries (in fact, this is where the term row-finite come from!), and each column of the transpose A_E^t contains a finite number of nonzero entries. Therefore, we have a map $A_E^t : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$ defined by left multiplication. (The column-finiteness of A_E^t ensures that $A_E^t x \in \bigoplus_{E^0} \mathbb{Z}$ for each $x \in \bigoplus_{E^0} \mathbb{Z}$.) If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family generating $C^*(E)$, and if we identify each $[p_v]_0$ with the element $\delta_v \in \bigoplus_{E^0} \mathbb{Z}$ containing a 1 in the v^{th} position and 0's elsewhere, then the relation in Remark 3.6 may be summarized as saying $(A_E^t - I)\delta_v$ is equivalent to 0 for all $v \in E^0$.

We are now in a position to describe how to compute the K-theory of the C^*-algebra of a row-finite graph with no sinks. This computation was first done in [35, Theorem 3.2]. The computation and its proof have also been discussed in [34, Chapter 7]. (In both cases sinks were allowed in the graphs, but for the sake of simplicity we state the result here for row-finite graphs without sinks.

Theorem 3.9 (K-theory for Graph Algebras: The Row-Finite, No Sinks Case)**.** *Let $E = (E^0, E^1, r, s)$ be a row-finite graph with no sinks. If A_E is the vertex matrix of E, and $A_E^t - I : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$ by left multiplication, then*

$$K_0(C^*(E)) \cong \operatorname{coker}(A_E^t - I)$$

via an isomorphism taking $[p_v]_0$ to $[\delta_v]$ for each $v \in E^0$, and

$$K_1(C^*(E)) \cong \ker(A_E^t - I).$$

Theorem 3.9 shows that to calculate the K-theory of $C^*(E)$ for a row-finite graph E with no sinks, we simply have to write down the matrix $A_E^t - I$, and then calculate the cokernel and kernel of $A_E^t - I$. When E has a finite number of vertices, this can be done very easily.

Remark 3.10 (Computing the Kernel and Cokernel of a Finite Matrix)*.* Let A be an $m \times n$ matrix with integer entries, and consider $A : \mathbb{Z}^n \to \mathbb{Z}^m$ by left multiplication. By performing elementary row and column operations to A, we may obtain an $m \times n$ matrix whose only nonzero entries are on the diagonal and of the form

$$D = \begin{pmatrix} d_1 & & & & \cdots & 0 \\ & \ddots & & & & \vdots \\ & & d_k & & & \\ & & & 0 & & \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix}$$

where $d_1, \ldots, d_k$ are nonzero integers with $k \leq \min\{m, n\}$. Warning: Remember that since we are viewing A as a map from the $\mathbb{Z}$-module $\mathbb{Z}^n$ into the $\mathbb{Z}$-module $\mathbb{Z}^m$, the allowed elementary row (resp. column) operations are: (1) adding an *integer* multiple of one row (resp. column) to another row (resp. column), (2) interchanging any two rows (resp. columns), (3) multiplying an row (resp. column) by the unit 1 or the unit -1.

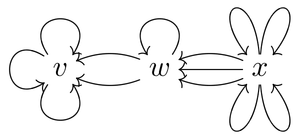
Since performing row operations to a matrix corresponds to postcomposing A with an automorphism on $\mathbb{Z}^m$ and performing column operations corresponds to precomposing A with an automorphism on $\mathbb{Z}^m$, we see that performing row and column operations will not change the isomorphism class of $\operatorname{coker} A$ or $\ker A$. Hence

$$\operatorname{coker} A \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m-k}$$

and

$$\ker A \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k}.$$

Example 3.11. Let E be the graph

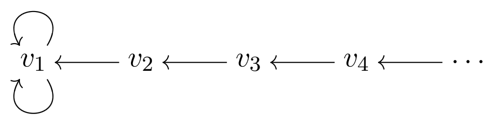


Then E is row-finite with no sinks, and the vertex matrix of this graph is $A_E = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix}$ and $A_E^t - I = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}$. One can perform elementary row and column operations on $A_E^t - I$ to obtain $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and therefore

$$K_0(C^*(G)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \qquad \text{and} \qquad K_1(C^*(G)) \cong \mathbb{Z}.$$

When a graph has an infinite number of vertices, the matrix $A_E^t - I$ will be infinite, so we cannot use the method described in Remark 3.10 to calculate the kernel and cokernel. However, in many situations, it is still possible to deduce what these groups are, as the following example shows.

Example 3.12. Let E be the graph



Then E is row-finite with no sinks, and the vertex matrix of this graph is $A_E = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$ and $A_E^t - I = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \\ 0 & 0 & -1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$. We see that an element $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \in \bigoplus_{i=1}^\infty \mathbb{Z}$ is in the kernel of $A_E^t - I$ if an only if the equations

$$x_1 + x_2 = 0, \quad x_2 - x_3 = 0, \quad x_3 - x_4 = 0, \quad \ldots$$

are satisfied. Since the x_i's are eventually zero this implies that $x_1 = x_2 = \ldots = 0$. Thus $K_1(C^*(E)) = \ker(A_E^t - I) = 0$. In addition, if $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} \in \bigoplus_{i=1}^\infty \mathbb{Z}$, then from some n we have $y_i = 0$ for $i \geq n$, and we see that

$$\vec{x} := \begin{pmatrix} y_1 + \ldots + y_n \\ -y_2 - \ldots - y_n \\ -y_3 - \ldots - y_n \\ \\ \vdots \\ -y_n \\ 0 \\ \\ \vdots \end{pmatrix} \in \bigoplus_{i=1}^\infty \mathbb{Z}.$$

Since $(A_E^t - I)\vec{x} = \vec{y}$ we see that $A_E^t - I$ is surjective, and $K_0(C^*(E)) = \operatorname{coker}(A_E^t - I) = 0$.

Having seen how to calculate K-theory in the case of a row-finite graph with no sinks, we now turn our attention to arbitrary graphs. In [39, Proposition 2] the K-theory computation for graph algebras was extended to non-row-finite graphs with a finite number of vertices. Additionally, in [8, Theorem 3.1] desingularization was used to extend Theorem 3.9 to all non-row-finite graphs. We present this result here.

Theorem 3.13 (K-theory for Graph Algebras: The General Case)**.** *Let $E = (E^0, E^1, r, s)$ be a graph. Also let J be the set vertices of E that are either sinks or infinite emitters, and let $I := E^0 \backslash J$. Then with respect to the decomposition $E^0 = I \cup J$ the vertex matrix of E will have the form*

$$A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have entries in $\mathbb{Z}$ and the $*$'s have entries in $\mathbb{Z} \cup \{\infty\}$. If we let $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$ by left multiplication, then

$$K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$$

via an isomorphism taking $[p_v]_0$ to $[\delta_v]$ for each $v \in E^0$, and

$$K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}.$$

Note that for a graph with a finite number of vertices, the matrix $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ will be finite and the method described in Remark 3.10 can be used to calculate the K-theory.

Example 3.14. Let E be the graph

$$\begin{tikzcd} v \arrow[r] \arrow[d] & w \arrow[loop right] \\ x \arrow[r, "(\infty)"'] & y \arrow[u, "(\infty)"'] \end{tikzcd}$$

Then x and y are infinite emitters, and $A_E = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \infty \\ 0 & \infty & 0 & 0 \end{pmatrix}$, so that $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Thus $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^4$. By performing elementary row and column operations to this matrix we obtain $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $K_0(C^*(E)) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \cong \mathbb{Z}^3$ and $K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \cong \mathbb{Z}$.

Remark 3.15 (The K-groups of graph algebras). Suppose that E is a graph. The calculation of K-theory described in Theorem 3.13 has the following implications.

- *For any graph E, the group $K_1(C^*(E))$ is free.* This follows from the fact that $\bigoplus_I \mathbb{Z}$ is a free group, and $K_1(C^*(E)) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ is a subgroup of this free group, and therefore also free. Remarkably, this is the only restriction on the K-theory; in fact, Szymański has shown in [40] that if G_0 and G_1 are countably generated abelian groups with G_1 free, then there exists a row-finite, transitive graph E such that $K_0(C^*(E)) \cong G_0$ and $K_1(C^*(E)) \cong G_1$. (Warning: In some of the graph algebra literature the word *free* has been mistakenly replaced by *torsion-free*. Recall that while these two notions are the same for finitely generated abelian groups, for countably generated abelian groups the free groups are a proper class of the torsion-free groups; for example, the additive group $\mathbb{Q}$ is a countably generated abelian group that is torsion-free but not free.)

- *If E is a finite graph with no sinks, then all the K-theory information of $C^*(E)$ is contained in $K_0(C^*(E))$.* In particular, the following hold:
  (1) the K-groups of $C^*(G)$ are finitely generated
  (2) $K_1(C^*(G))$ is a free group
  (3) $K_0(C^*(G)) \cong T \oplus K_1(C^*(G))$ for some finite torsion group T
  Consequently, if E_1 and E_2 are finite graphs, then $K_0(C^*(E_1)) \cong K_0(C^*(E_2))$ implies that $K_1(C^*(E_1)) \cong K_1(C^*(E_2))$.
- *If E is a graph that has a finite number of vertices (but possibly an infinite number of edges), then* $\operatorname{rank} K_0(C^*(E)) \geq \operatorname{rank} K_1(C^*(E))$. The reason for this is that Theorem 3.13 gives the short exact sequence:
  $$0 \longrightarrow K_1(C^*(G)) \longrightarrow \mathbb{Z}^I \longrightarrow \mathbb{Z}^I \oplus \mathbb{Z}^J \longrightarrow K_0(C^*(G)) \longrightarrow 0$$
  and since I and J are finite we have $\operatorname{rank} K_0(C^*(G)) \geq \operatorname{rank} K_1(C^*(G))$.
- *If E is a graph in which every vertex is either a sink or an infinite emitter, then $K_0(C^*(E)) \cong \bigoplus_{E^0} \mathbb{Z}$ and $K_1(C^*(E)) \cong 0$.* This is because the set I described in Theorem 3.13 is empty so $\bigoplus_I \mathbb{Z} = 0$.

In addition to the vertex matrix, one may also use the edge matrix to calculate the K-theory of a graph algebra.

Definition 3.16. Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sinks. The *edge matrix* of E is the (possibly infinite) $E^1 \times E^1$ matrix B_E whose entries are

$$B_E(e,f) := \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

Note that if E is row-finite, then any row of B_E will have at most a finite number of nonzero entries. Thus we have a well-defined map $B_E^t - I : \bigoplus_{E^1} \mathbb{Z} \to \bigoplus_{E^1} \mathbb{Z}$ given by left multiplication.

Proposition 3.17. *If E is a row-finite graph with no sinks, and we let $A_E^t - I : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z}$ and $B_E^t - I : \bigoplus_{E^1} \mathbb{Z} \to \bigoplus_{E^1} \mathbb{Z}$ by left multiplication, then*

$$\operatorname{coker}(A_E^t - I) \cong \operatorname{coker}(B_E^t - I) \quad \textit{and} \quad \ker(A_E^t - I) \cong \ker(B_E^t - I).$$

Proof. Let S denote the $E^0 \times E^1$ matrix defined by

$$S(v,e) := \begin{cases} 1 & \text{if } s(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Also let R denote the $E^1 \times E^0$ matrix defined by

$$R(e,v) := \begin{cases} 1 & \text{if } r(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that

$$SR = A_E \quad \text{and} \quad RS = B_E$$

and

$$R^t S^t = A_E^t \quad \text{and} \quad S^t R^t = B_E^t. \tag{3.1}$$

We define a map from $\operatorname{coker}(A_E^t - I) \to \operatorname{coker}(B_E^t - I)$ by $x + \operatorname{im}(A_E^t - I) \mapsto S^t x + \operatorname{im}(B_E^t - I)$. This is well-defined because if $x + \operatorname{im}(A_E^t - I) = y + \operatorname{im}(A_E^t - I)$, then $x - y = (A_E^t - I)z$ for some z and

$$S^t x - S^t y = S^t(x-y) = S^t(A_E^t - I)z = (B_E^t S^t - S^t)z = (B^t - I)S^t z$$

so $S^t x + \operatorname{im}(B_E^t - I) = S^t y + \operatorname{im}(B_E^t - I)$. In a similar manner we may define a map from $\operatorname{coker}(B_E^t - I) \to \operatorname{coker}(A_E^t - I)$ by $x + \operatorname{im}(B_E^t - I) \mapsto R^t x + \operatorname{im}(A_E^t - I)$. We see that these maps are inverses of each other because

$$\begin{aligned} R^t S^t x + \operatorname{im}(A_E^t - I) &= A_E^t x + \operatorname{im}(A_E^t - I) \\ &= [x + (A_E^t - I)x] + \operatorname{im}(A_E^t - I) = x + \operatorname{im}(A_E^t - I) \end{aligned}$$

and

$$\begin{aligned} S^t R^t x + \operatorname{im}(B_E^t - I) &= B_E^t x + \operatorname{im}(B_E^t - I) \\ &= [x + (B_E^t - I)x] + \operatorname{im}(B_E^t - I) = x + \operatorname{im}(B_E^t - I). \end{aligned}$$

Thus $\operatorname{coker}(A_E^t - I) \cong \operatorname{coker}(B_E^t - I)$.

In addition, we may define a map from $\ker(A_E^t - I)$ to $\ker(B_E^t - I)$ by $x \mapsto S^t x$. We see that this map takes values in $\ker(B_E^t - I)$, because if $x \in \ker(A^t - I)$ then

$$(B_E^t - I)S^t x = (S^t A_E^t - S^t)x = S^t(A_E^t - I)x = 0.$$

Similarly, we may define a map from $\ker(B_E^t - I)$ to $\ker(A_E^t - I)$ by $x \mapsto R^t x$. We see that these maps are inverses of each other because if $x \in \ker(A_E^t - I)$, then

$$R^t S^t x = A_E^t x = x + (A_E^t - I)x = x$$

and if $x \in \ker(B_E^t - I)$, then

$$S^t R^t x = B_E^t x = x + (B_E^t - I)x = x.$$

Thus $\ker(A_E^t - I) \cong \ker(B_E^t - I)$. □

The above proposition together with Theorem 3.9 gives the following.

Proposition 3.18. *Let $E = (E^0, E^1, r, s)$ be a row-finite graph with no sinks. If B_E is the edge matrix of E, and $B_E^t - I : \bigoplus_{E^1} \mathbb{Z} \to \bigoplus_{E^1} \mathbb{Z}$ by left multiplication, then*

$$K_0(C^*(E)) \cong \operatorname{coker}(B_E^t - I)$$

via an isomorphism taking $[p_v]_0$ to $\sum_{s(e)=v} [\delta_e]$ for each $v \in E^0$, and

$$K_1(C^*(E)) \cong \ker(B_E^t - I).$$

Remark 3.19. Theorem 3.9 and Proposition 3.18 show that when E is a row-finite graph with no sinks, then to calculate the K-theory of $C^*(E)$ we may use either the vertex matrix A_E or the edge matrix B_E. In certain situations, one of these matrices may be easier to use than the following: The edge matrix has the advantage that all its entries are in $\{0, 1\}$ and this simplifies row and column operations. However, the edge matrix has the disadvantage that it is typically much larger than the vertex matrix.

Remark 3.20. In addition to computing the isomorphism classes of the K-groups of a graph algebra, one often wants to compute the ordering on $K_0(C^*(E))$. When E is a row-finite graph (possibly with sinks) it follows from [45, Lemma 2.1] that if E is row-finite (but possibly has sinks), then the isomorphism described in Theorem 3.13 takes $K_0(C^*(E))_+$ to $\{[x] : x \in \bigoplus_I \mathbb{N} \oplus \bigoplus_J \mathbb{N}\}$, where $[x]$ denotes the class of x in coker $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$. If E is not row-finite, then [45, Theorem 2.2] shows that the isomorphism described in Theorem 3.13 takes $K_0(C^*(E))_+$ onto the semigroup of coker $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ generated by the set

$$\{[\delta_v] : v \in E^0\} \cup \{[\delta_v] - \sum_{e \in S} [\delta_{r(e)}] : v \text{ is an infinite emitter and } S \text{ is a finite subset of } s^{-1}(v)\}.$$

Remark 3.21. We conclude by commenting that Ext, the dual theory for K-theory, has also been computed for C^*-algebras of graph satisfying Condition (L). This was done for row-finite graph algebras in [42, Theorem 5.16] and arbitrary graph algebras in [8, Theorem 3.1]. Specifically, if $A_E = \left(\begin{smallmatrix} B & C \\ * & * \end{smallmatrix}\right)$ is the decomposition described in Theorem 3.13, then $(B - I \ C) : \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \to \prod_I \mathbb{Z}$ defines a mapping by left multiplication and $\operatorname{Ext}(C^*(E)) \cong \operatorname{coker}(B - I \ C)$. (Note that the domain and codomain of this map involve direct products rather than direct sums.)

3.2. **Classification Theorems.** One wants to calculate the K-theory of a C^*-algebra because it provides an invariant. K-theory can always be used to tell if two C^*-algebras are different: If two C^*-algebras have non-isomorphic K-theory, for example, then those C^*-algebras are not Morita equivalent (and hence also not isomorphic). More importantly, in certain situations K-theory can be used to tell when C^*-algebras are the same.

Elliott has conjectured that all nuclear C^*-algebras can be classified by K-theoretic information, which is now called the *Elliott invariant.* (For a general nuclear C^*-algebra the Elliott invariant involves K-theoretic information beyond the K_0 and K_1 groups that we have not discussed. However, for the classes of C^*-algebras we consider, the invariant will only involve the ordered K_0-group and the K_1-group.) The Elliott conjecture has been verified in a number of special cases, including AF-algebras and certain simple purely infinite C^*-algebras. Using these results we shall describe how all simple graph algebras are classified by their K-theory, and we shall give an algorithm for determining whether two simple graph algebras are isomorphic and whether they are Morita equivalent.

3.2.1. *AF-algebras.* If A is an AF-algebra, then $K_1(A) = 0$. Hence all the K-theoretic information of A is contained in the group $K_0(A)$. The AF-algebras were one of the first class of C^*-algebras to be classified by K-theory, and this was done by Elliott in the 1970's [10]. It was this success that inspired Elliott to conjecture that wider classes of C^*-algebras can be classified by K-theoretic information.

The following theorem appears in most books on operator algebra K-theory; see, for instance, [47, Theorem 12.1.3].

Theorem 3.22 (Elliott's Theorem)**.** *Let A and B be AF-algebras. Then A and B are Morita equivalent if and only if $(K_0(A), K_0(A)_+) \cong (K_0(B), K_0(B)_+)$. That is, the ordered K_0-group is a complete Morita equivalence invariant for AF-algebras.*

If A and B are both unital, then A and B are isomorphic if and only if $(K_0(A), K_0(A)_+, [1_A]_0) \cong (K_0(B), K_0(B)_+, [1_B]_0)$. That is, the ordered K_0-group together with the position of the unit is a complete isomorphism invariant for AF-algebras.

Remark 3.23. If E is a graph with no loops, then as described in Remark 1.26, $C^*(E)$ is an AF-algebra. Using Theorem 3.13 and Remark 3.20, we can calculate $(K_0(C^*(E)), K_0(C^*(E))_+)$ and determine the Morita equivalence class of $C^*(E)$.

Remark 3.24. Although Theorem 3.22 only talks of the Morita equivalence class of nonunital AF-algebras, the isomorphism class of a nonunital AF-algebra is also determined by K-theoretic information. As described in [47, Theorem 12.1.3] if A is an AF-algebra, then the scaled ordered group $(K_0(A), K_0(A)_+, D(A))$ is a complete isomorphism invariant of A. We have not discussed the *scale* $D(A) := \{[p]_0 : p \in \text{Proj}(A)\}$ of the K_0-group because the author does not know of an easy way to calculate it for graph algebras, and so it does not fit easily into our current discussion. However, when A is unital, the scale $D(A)$ may be replaced by the position of the class of the unit in the K_0, as described in our statement of Theorem 3.22.

3.2.2. *Kirchberg-Phillips Algebras.* In addition to AF-algebras, certain simple purely infinite C^*-algebras have been classified by their K-theory. This result is known as the Kirchberg-Phillips Classification Theorem, and was proven independently by Kirchberg and Phillips using different methods. Phillips' result appears in [32]. Kirchberg's version is not yet published, but a preliminary account, including proofs of his "Geneva Theorems" and partial proofs of his version of the classification theorem, was circulated in 1994.

Theorem 3.25. *Let A and B be purely infinite, simple, separable, nuclear C^*-algebras that satisfy the Universal Coefficients Theorem.*

(1) *If A and B are both unital, then A is isomorphic to B if and only if $(K_0(A), [1_A]_0) \cong (K_0(B), [1_B]_0)$ and $K_1(A) \cong K_1(B)$.*
(2) *If A and B are nonunital, then A is isomorphic to B if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.*

Remark 3.26. Let $\mathcal{K}$ denote the compact operators on a separable infinite-dimensional Hilbert space. We say that a C^*-algebra is *stable* if $A \otimes \mathcal{K} \cong A$. For any C^*-algebra, we see that $A \otimes \mathcal{K}$ will be stable because $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$. We call $A \otimes \mathcal{K}$ the *stabilization of* A. The stabilization of a C^*-algebra is always nonunital, and both pure infiniteness and AF-ness are preserved by stabilization. In addition, $K_0(A \otimes \mathcal{K}) \cong K_0(A)$ and $K_1(A \otimes \mathcal{K}) \cong K_1(A)$.

The Brown-Green-Rieffel Theorem asserts that two separable C^*-algebras A and B are Morita equivalent if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. Furthermore, Zhang's dichotomy [48] says that all separable, nonunital purely infinite C^*-algebras are stable, and thus all separable, nonunital purely infinite C^*-algebras are Morita Equivalent if and only if they are isomorphic.

Using these facts, the Kirchberg-Phillips Classification Theorem gives the following.

Corollary 3.27. *Let A and B be purely infinite, simple, separable, nuclear C^*-algebras that satisfy the Universal Coefficients Theorem. Then three cases can occur.*

Case 1: A and B are both unital.

Then A and B are isomorphic if and only if $(K_0(A), [1_A]_0) \cong (K_0(B), [1_B]_0)$ and $K_1(A) \cong K_1(B)$. In addition, A and B are Morita equivalent if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$, and in this case $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.

Case 2: A and B are both nonunital.

Then A and B are isomorphic if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. In addition, A and B are Morita equivalent if and only if A and B are isomorphic.

Case 3: One of A and B is nonunital and the other is unital.

Suppose A is nonunital and B is unital. Then A and B are not isomorphic, and A and B are Morita equivalent if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$, in which case $A \cong B \otimes \mathcal{K}$.

3.2.3. *Graph Algebras.* To apply these classifications to graph algebras, we first consider when a graph algebra will satisfy the hypotheses of Theorem 3.25. To begin, we see that all graph algebras are separable since they are generated by the countable collection $\{s_e, p_v : e \in E^1, v \in E^0\}$. In addition, it is shown in [26, Proposition 2.6] that for any directed graph E the crossed product $C^*(E) \times_\alpha \mathbb{T}$ is an AF-algebra. (The proof in [26] is for row-finite graphs, but it should hold for arbitrary graphs as well.) Therefore from the Takesaki-Takai duality theorem (see [31, Theorem 7.9.3]) one has

$$C^*(E) \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (C^*(E) \times_\alpha \mathbb{T}) \times_{\hat{\alpha}} \mathbb{Z}$$

and hence $C^*(E)$ is stably isomorphic to the crossed product of an AF-algebra by $\mathbb{Z}$. It then follows from [4, Corollary 3.2] and [5, Proposition 6.8] that $C^*(E)$ is nuclear, and it follows from [38, Theorem 1.17] and [3, Chapter 23] that $C^*(E)$ satisfies the UCT. Hence the Kirchberg-Phillips Classification Theorem applies to any purely infinite simple graph algebra.

Moreover, if $E = (E^0, E^1, r, s)$ is a graph, then $C^*(E)$ is unital if and only if E^0 is finite. When E^0 is finite, one can easily check that the Cuntz-Krieger relations imply that $1 = \sum_{v \in E^0} p_v$ is a unit for $C^*(E)$. Note that the isomorphisms in Theorem 3.9 and Theorem 3.13 take $[1]_0$ to the element $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}\right]$ in the appropriate cokernel.

Since the dichotomy for simple graph algebras given in Proposition 2.15 implies that all simple graph algebras are either AF or purely infinite (depending on whether or not the graph has a loop), we may use Theorem 3.22, Theorem 3.25, and Corollary 3.27 to classify simple graph algebras. We summarize the implications of these results here.

Theorem 3.28 (Classification of Simple Graph Algebras)**.** *Let E and F be graphs, and suppose that $C^*(E)$ and $C^*(F)$ are simple (characterized for row-finite graphs in Theorem 1.23 and for arbitrary graphs in Theorem 2.12). Then there are three possible cases.*

CASE 1: BOTH E AND F HAVE NO LOOPS.

Then $C^*(E)$ and $C^*(F)$ are AF, and $C^*(E)$ and $C^*(F)$ are Morita equivalent if and only if $(K_0(C^*(E)), K_0(C^*(E))_+) \cong (K_0(C^*(F)), K_0(C^*(F))_+)$, in which case $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$.

Furthermore, if A and B are unital, then $C^*(E) \cong C^*(F)$ if and only if $(K_0(C^*(E)), K_0(C^*(E))_+, [1_{C^*(E)}]_0) \cong (K_0(C^*(F)), K_0(C^*(F))_+, [1_{C^*(F)}]_0)$.

CASE 2: BOTH E AND F EACH HAVE AT LEAST ONE LOOP.

Then $C^*(E)$ and $C^*(F)$ are purely infinite and there are three subcases.

(i) *If E^0 and F^0 are both finite, then $C^*(E) \cong C^*(F)$ if and only if $(K_0(C^*(E)), [1_{C^*(E)}]_0) \cong (K_0(C^*(F)), [1_{C^*(F)}]_0)$ and $K_1(C^*(E)) \cong K_1(C^*(F))$. Furthermore, $C^*(E)$ and $C^*(F)$ are Morita equivalent if and only if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F))$, in which case $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$.*

(ii) *If E^0 and F^0 are both infinite, then $C^*(E) \cong C^*(F)$ if and only if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F))$. In addition, $C^*(E)$ and $C^*(F)$ are isomorphic if and only if $C^*(E)$ and $C^*(F)$ are Morita equivalent.*

(iii) *If one of E^0 and F^0 is infinite and the other is finite (let us say E^0 is infinite and F^0 is finite), then $C^*(E)$ and $C^*(F)$ are not isomorphic. In addition $C^*(E)$ and $C^*(F)$ are Morita equivalent if and only if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F))$ in which case $C^*(E) \cong C^*(F) \otimes \mathcal{K}$.*

CASE 3: ONE OF E AND F HAS AT LEAST ONE LOOP AND THE OTHER HAS NO LOOPS.

Then one of $C^*(E)$ and $C^*(F)$ is purely infinite while the other is an AF-algebra. Hence $C^*(E)$ and $C^*(F)$ are not Morita equivalent (and therefore also not isomorphic).

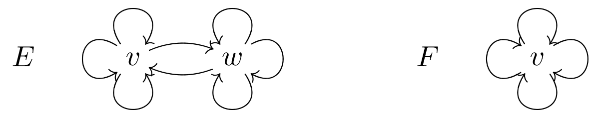
Remark 3.29. Notice that in Case 1 we did not give sufficient conditions for $C^*(E)$ and $C^*(F)$ to be isomorphic when $C^*(E)$ and $C^*(F)$ are nonunital. This is the only case missing from the above theorem, and if we were able to describe the scale of the K_0-group of a graph algebra, then as described in Remark 3.24 we would have a complete description.

When calculating the K-theory of a unital graph algebra $C^*(E)$, we need to calculate the kernel and cokernel of the finite matrix $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$, but in addition, we need to keep track of the position of the unit $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}\right]$.

Remark 3.30 (Computing the Position of the Unit). This remark is a follow-up to Remark 3.10. If A is an $m \times n$ matrix and $A : \mathbb{Z}^n \to Z^m$ by left multiplication, then as described in Remark 3.10 we may perform elementary row and column operations on A to form a matrix D whose only nonzero entries $d_1, \ldots, d_k$ are on the diagonal. Since performing elementary row operations (resp. column operations) corresponds to multiplying A on the left (resp. right) by an elementary matrix, by keeping track of the row and column operations, we may write $MAN = D$ for some invertible matrices M and N. Since M and N are invertible, we see that $\ker A \cong \ker D = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n-k}$. We also see that $\operatorname{coker} A \cong \operatorname{coker} D = \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \mathbb{Z}/d_k\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m-k}$ and that this isomorphism takes the class $x + \operatorname{im} A \in \operatorname{coker} A$ to the class $M^{-1}x + \operatorname{im} D \in \operatorname{coker} D$.

We now consider some examples which make use of Theorem 3.28.

Example 3.31. Let E and F be the following graphs.



Then $C^*(E)$ and $C^*(F)$ are simple and purely infinite, and the graphs fall into Case 2(i) of Theorem 3.28. We see that $A_F = (4)$, and

$$A_F^t - I = (3) : \mathbb{Z} \to \mathbb{Z}.$$

Thus we have $(K_0(C^*(F)), [1_{C^*(F)}]_0) \cong (\mathbb{Z}_3, [1])$ and $K_1(C^*(E)) = 0$.

Furthermore, $A_E = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $A_E^t - I = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2$. We shall now reduce this matrix to a diagonal matrix, keeping track of the row and column operations that we use.

$$\begin{aligned}
\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix} & \qquad (-2 \text{ times Row 1 added to Row 2}) \\
\sim \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} & \qquad (-2 \text{ times Column 2 added to Column 1}) \\
\sim \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} & \qquad (\text{Exchange Row 1 and Row 2}) \\
\sim \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} & \qquad (-1 \text{ times Column 1})
\end{aligned}$$

Since row operations correspond to multiplying on the left by the associated elementary matrices, and column operations correspond to multiplying on the right by the associated elementary matrices, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

and if we let $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$, and $N = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, then $M \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} N = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus we see that $\operatorname{coker}(A_E^t - I) \cong \mathbb{Z}_3 \oplus 0$ and since $M^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ we have

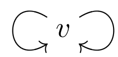
$$M^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \operatorname{im} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \operatorname{im} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \operatorname{im} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $(K_0(C^*(E)), [1_{C^*(E)}]_0) \cong (\mathbb{Z}_3, [1])$ and $K_1(C^*(E)) = 0$.

It follows that $C^*(E) \cong C^*(F)$.

Remark 3.32. Notice that although we were able to determine that $C^*(E) \cong C^*(F)$ in Example 3.31, we have no idea what the isomorphism is (since the Kirchberg-Phillips Classification Theorem only tells us of the existence of the isomorphism between C^*-algebras). It would be interesting, and possibly difficult, to show how in situations such as this one can exhibit a Cuntz-Krieger E-family in $C^*(F)$, so that the isomorphism may be described explicitly.

Example 3.33. Let E be the graph in Example 3.12, and let F be the graph



Then $C^*(E)$ and $C^*(F)$ are simple and purely infinite, and the graphs fall into Case 2(iii) of Theorem 3.28. We see that $A_F = (2)$, and

$$A_F^t - I = (1) : \mathbb{Z} \to \mathbb{Z}.$$

Thus we have $K_0(C^*(F)) = 0$ and $K_1(C^*(F)) = 0$. Since it was shown in Example 3.12 that $K_0(C^*(E)) = 0$ and $K_1(C^*(E)) = 0$, we have that $C^*(F) \cong C^*(E) \otimes \mathcal{K}$.

Moreover, since we know that $C^*(E)$ is the Cuntz algebra $\mathcal{O}_2$, we have that $C^*(F) \cong \mathcal{O}_2 \otimes \mathcal{K}$ so that $C^*(F)$ is the stabilization of $\mathcal{O}_2$.

We conclude this section by discussing which AF-algebras and which Kirchberg-Phillips algebras arise as graph algebras.

It was shown in [6] and [46] that every AF-algebra is Morita equivalent to a graph algebra. In addition, it is known that there are AF-algebras that are not isomorphic to any graph algebra.

With regards to the Kirchberg-Phillips algebras, Szymański has proven the following in [40, Theorem 1.2].

Theorem 3.34. *Let G_0 and G_1 be countable abelian groups with G_1 free. If $g \in G_0$, then there is a row-finite, transitive graph E with an infinite number of vertices, and a vertex $v \in E^0$ such that $(K_0(C^*(E)), [p_v]_0) \cong (G_0, g)$ and $K_1(C^*(E)) \cong G_1$.*

The proof of Szymański's theorem is very concrete, and in fact he describes how to construct the graph E from (G_0, g) and G_1. Also, note that if E is a row-finite, transitive graph with an infinite number of vertices, then $C^*(E)$ is simple, purely infinite, and nonunital.

The Kirchberg-Phillips Classification Theorem then gives the following two corollaries.

Corollary 3.35. *Let A be a purely infinite, simple, separable, nonunital, nuclear C^*-algebra that satisfies the Universal Coefficient Theorem. If $K_1(A)$ is free, then A is isomorphic to the C^*-algebra of a row-finite, transitive graph.*

Corollary 3.36. *Let A be a purely infinite, simple, separable, unital, nuclear C^*-algebra that satisfies the Universal Coefficient Theorem. If $K_1(A)$ is* *free, then A is isomorphic to a full corner of the C^*-algebra of a row-finite, transitive graph.*

Proof. Choose a row-finite, transitive graph E and a vertex v such that $(K_0(C^*(E)), [p_v]_0) \cong (K_0(A), [1_A]_0)$ and $K_1(C^*(E)) \cong K_1(A)$. If we consider the corner $p_vC^*(E)p_v$, then since $C^*(E)$ is simple, we see that this corner is full. Hence $p_vC^*(E)p_v$ is Morita equivalent to $C^*(E)$, and $p_vC^*(E)p_v$ is purely infinite, simple, separable, nuclear, and satisfies the Universal Coefficient Theorem. Furthermore, the projection p_v is a unit for $p_vC^*(E)p_v$, and since the inclusion of $p_vC^*(E)p_v$ into $C^*(E)$ preserves K-theory [30, Proposition 1.2], we have that $(K_0(p_vC^*(E)p_v), [p_v]_0) \cong (K_0(A), [1_A]_0)$ and $K_1(p_vC^*(E)p_v) \cong K_1(A)$. Thus $p_vC^*(E)p_v \cong A$. □

4. Generalizations of Graph Algebras

Since the introduction of graph algebras, various authors have considered a myriad of generalizations in which a C^*-algebra is associated to an object other than a directed graph. In particular generalizations this object may be a matrix, a Hilbert C^*-module, or something more exotic. The goal in these generalizations is to produce a class of C^*-algebras with the following properties:

(1) the class includes graph algebras in a natural way, as well as C^*-algebras that are not graph algebras; and
(2) for each C^*-algebra in the class, the structure of the C^*-algebra is reflected in the object from which it is created.

In this section, we will discuss some of the generalizations which have become prominent in the literature in the past few years. Because each of these classes has been the subject of many papers, a complete description of each of the theories and their developments is beyond our scope. Instead, we will simply attempt a whirlwind survey of a handful of important classes. In each case, our goal will be to

(1) define the basic objects that will be used in place of directed graphs, and discuss how a C^*-algebra can be constructed from such an object,
(2) explain how graph algebras are special cases of these C^*-algebras, and
(3) compare and contrast the theory for these C^*-algebras to the theory for graph C^*-algebras.

We will consider four classes of C^*-algebras that generalize graph algebras: Exel-Laca algebras, ultragraph algebras, Cuntz-Pimsner algebras, and topological quiver algebras.

4.1. **Exel-Laca Algebras.**

The Exel-Laca algebras, which were introduced in [12], can be thought of as Cuntz-Krieger algebras for infinite matrices. The idea is that one begins with a countable square matrix A with entries in $\{0, 1\}$. One then defines the Exel-Laca algebra $\mathcal{O}_A$ to be the C^*-algebra generated by partial isometries (one for each row) satisfying relations determined by A. These relations are meant to generalize the relations used to define a Cuntz-Krieger algebra (and when A is finite, they reduce to precisely these relations). The difficulty comes in defining the relations when A is not row-finite.

Definition 4.1 (Exel-Laca). Let I be a countable set and let $A = \{A(i,j)_{i,j\in I}\}$ be a $\{0,1\}$-matrix over I with no identically zero rows. The Exel-Laca algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $\{s_i : i \in I\}$ with commuting initial projections and mutually orthogonal range projections satisfying $s_i^* s_i s_j s_j^* = A(i,j) s_j s_j^*$ and

$$\prod_{x\in X} s_x^* s_x \prod_{y\in Y} (1 - s_y^* s_y) = \sum_{j\in I} A(X,Y,j) s_j s_j^* \tag{4.1}$$

whenever X and Y are finite subsets of I such that the function

$$j \in I \mapsto A(X,Y,j) := \prod_{x\in X} A(x,j) \prod_{y\in Y} (1 - A(y,j))$$

is finitely supported.

To understand where this last relation comes from, notice that combinations of formal infinite sums obtained from the original Cuntz-Krieger relations could give relations involving finite sums, and (4.1) says that these finite relations must be satisfied in $\mathcal{O}_A$; see the introduction of [12] for more details.

Although there is reference to a unit in (4.1), this relation applies to algebras that are not necessarily unital, with the convention that if a 1 still appears after expanding the product in (4.1), then the relation implicitly states that $\mathcal{O}_A$ is unital. It is also important to realize that the relation (4.1) also applies when the function $j \mapsto A(X,Y,j)$ is identically zero. This particular instance of (4.1) is interesting in itself so we emphasize it by stating the associated relation separately:

$$\prod_{x\in X} s_x^* s_x \prod_{y\in Y} (1 - s_y^* s_y) = 0 \tag{4.2}$$

whenever X and Y are finite subsets of I such that $A(X,Y,j) = 0$ for every $j \in I$.

Remark 4.2. If E is a graph with no sinks or sources, then $C^*(E)$ is an Exel-Laca algebra. In fact, it is shown in [14, Proposition 9] that if E has no sinks or sources, and if $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family, then $\{s_e : e \in E^1\}$ is a collection of partial isometries satisfying the relations defining $\mathcal{O}_{B_E}$, where B_E is the edge matrix of E.

Not all graph algebras are Exel-Laca algebras; there are examples of graphs with sinks, and other examples of graphs with sources, whose C^*-algebras are not isomorphic to any Exel-Laca algebra.

There is a Cuntz-Krieger Uniqueness Theorem for Exel-Laca algebras. If A is a countable square matrix over I with entries in $\{0,1\}$, then we define a directed graph $\mathrm{Gr}(A)$, by letting the vertices of this graph be I, and then drawing an edge from i to j if and only if $A(i,j)=1$.

The following theorem is an equivalent reformulation of [12, Theorem 13.1].

Theorem 4.3 (Cuntz-Krieger Uniqueness Theorem)**.** *Let I be a countable set and let $A = \{A(i,j)_{i,j\in I}\}$ be a $\{0,1\}$-matrix over I with no identically zero rows. If* $\mathrm{Gr}(A)$ *satisfies Condition (L), and if $\rho : \mathcal{O}_A \to B$ is a $*$-homomorphism between C^*-algebras with the property that $\rho(S_i) \neq 0$ for all $i \in I$, then ρ is injective.*

The graph $\mathrm{Gr}(A)$ is also useful in describing pure infiniteness of Exel-Laca algebras. It is shown in [12, Theorem 16.2] that every nonzero hereditary subalgebra of $\mathcal{O}_A$ contains an infinite projection if and only if $\mathrm{Gr}(A)$ satisfies Condition (L) and every vertex in $\mathrm{Gr}(A)$ can reach a loop in $\mathrm{Gr}(A)$.

Simplicity for Exel-Laca algebras is more complicated. Exel and Laca showed in [12, Theorem 14.1] that if $\mathrm{Gr}(A)$ is transitive and not a single loop, then $\mathcal{O}_A$ is simple. A complete characterization of simplicity was obtained by Szymański in [41], where he defined a notion of saturated hereditary subset for A, and proved that $\mathcal{O}_A$ is simple if and only if $\mathrm{Gr}(A)$ satisfies Condition (L) and A has no proper nontrivial saturated hereditary subsets. (We mention that there are examples of a matrix A such that $\mathcal{O}_A$ is simple, but $C^*(\mathrm{Gr}(A))$ is not simple!) Szymański's result can also be used to show that the dichotomy holds for simple Exel-Laca algebras: every simple Exel-Laca algebra is either AF or purely infinite.

In addition, the universal property of $\mathcal{O}_A$ gives a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut}\mathcal{O}_A$ with $\gamma_z(S_i) = zS_i$, and there is a gauge-invariant uniqueness theorem for Exel-Laca algebras. Exel and Laca also calculate the K-theory of $\mathcal{O}_A$ in [13].

4.2. **Ultragraph Algebras.** One difficulty with Exel-Laca algebras is that the matrix A lacks the visual appeal one finds in a graph. In fact when describing appropriate version of graph notions, such as Condition (L) or vertices being able to reach loops, one introduces an associated graph $\mathrm{Gr}(A)$. However, as we saw in our description of simplicity, the graph $\mathrm{Gr}(A)$ does not fully reflect the structure of $\mathcal{O}_A$. In addition, when working with Exel-Laca algebras one must deal with complicated relations among generators, such as in (4.1), which again lack the visual appeal of graph algebras.

An attempt to study Exel-Laca algebras using a generalized notion of a graph, called an "ultragraph", was undertaken in [43] and [44]. Roughly speaking, an ultragraph is a directed graph in which the range of an edge is allowed to be a set of vertices rather than a single vertex.

Definition 4.4. An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and

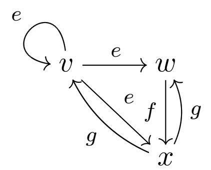
$r : \mathcal{G}^1 \to P(G^0)$, where $P(G^0)$ denotes the collection of nonempty subsets of G^0.

Remark 4.5. Note that a graph may be viewed as a special type of ultragraph in which $r(e)$ is a singleton set for each edge e.

Example 4.6. A convenient way to draw ultragraphs is to first draw the set G^0 of vertices, and then for each edge $e \in \mathcal{G}^1$ draw an arrow labeled e from $s(e)$ to each vertex in $r(e)$. For instance, the ultragraph given by

$$\begin{array}{llll} G^0 = \{v, w, x\} & s(e) = v & s(f) = w & s(g) = x \\ \mathcal{G}^1 = \{e, f, g\} & r(e) = \{v, w, x\} & r(f) = \{x\} & r(g) = \{v, w\} \end{array}$$

may be drawn as



We then identify any arrows with the same label, thinking of them as being a single edge. Thus in the above example there are only three edges, e, f, and g, despite the fact that there are six arrows drawn.

A vertex $v \in G^0$ is called a *sink* if $|s^{-1}(v)| = 0$ and an *infinite emitter* if $|s^{-1}(v)| = \infty$.

For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ we let $\mathcal{G}^0$ denote the smallest subcollection of the power set of G^0 that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite intersections and finite unions. Roughly speaking, the elements of $\{v : v \in G^0\} \cup \{r(e) : e \in \mathcal{G}^1\}$ play the role of "generalized vertices" and $\mathcal{G}^0$ plays the role of "subsets of generalized vertices".

Definition 4.7. If $\mathcal{G}$ is an ultragraph, a *Cuntz-Krieger $\mathcal{G}$-family* is a collection of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges and a collection of projections $\{p_A : A \in \mathcal{G}^0\}$ that satisfy

(1) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$
(2) $s_e^* s_e = p_{r(e)}$ for all $e \in \mathcal{G}^1$
(3) $s_e s_e^* \leq p_{s(e)}$ for all $e \in \mathcal{G}^1$
(4) $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.

We define $C^*(\mathcal{G})$ to be the C^*-algebra generated by a universal Cuntz-Krieger $\mathcal{G}$-family. When A is a singleton set $\{v\}$, we write p_v in place of $p_{\{v\}}$.

When $\mathcal{G}$ has the property that $r(e)$ is a singleton set for every edge e, then $\mathcal{G}$ may be viewed as a graph (and, in fact, every graph arises this way). In this case $\mathcal{G}^0$ is simply the finite subsets of G^0, and if $\{s_e, p_v\}$ is a Cuntz-Krieger family for the graph algebra associated to $\mathcal{G}$, then by defining $p_A := \sum_{v \in A} p_v$ we see that $\{p_A, s_e\}$ is a Cuntz-Krieger $\mathcal{G}$-family, and thus the graph algebra and the ultragraph algebra for $\mathcal{G}$ coincide. (The details of this argument are carried out in [43, Proposition 3.1].)

When $\mathcal{G}$ is an ultragraph with an edge e such that $r(e)$ is an infinite set, then the projection $p_{r(e)}$ will dominate p_v for all $v \in r(e)$, but $p_{r(e)}$ will not be the sum of any finite collection of p_v's. It is projections such as these that allow for ultragraph algebras that are not graph algebras.

In addition to containing all graph algebras, it is shown in [43, §4] that all Exel-Laca algebras are ultragraphs. Furthermore, it is shown in [44] that there are ultragraph algebras that are neither Exel-Laca algebras nor graph algebras. Thus the ultragraph algebras provide us with a strictly larger class than graph algebras and Exel-Laca algebras.

A *path* in an ultragraph $\mathcal{G}$ is a sequence of edges $\alpha_1 \ldots \alpha_n$ with $s(\alpha_i) \in r(\alpha_{i-1})$ for $i = 2, 3, \ldots, n$

Definition 4.8. If $\mathcal{G}$ is an ultragraph, then a *loop* is a path $\alpha_1 \ldots \alpha_n$ with $s(\alpha_1) \in r(\alpha_n)$. An *exit* for a loop is either of the following:

(1) an edge $e \in \mathcal{G}^1$ such that there exists an i for which $s(e) \in r(\alpha_i)$ but $e \neq \alpha_{i+1}$
(2) a sink w such that $w \in r(\alpha_i)$ for some i.

An exit for a loop is simply something (an edge or sink) that allows one to avoid repeating the same sequence $\alpha_1 \ldots \alpha_n$ as one follows edges in $\mathcal{G}$. Also note that if $\alpha_1 \ldots \alpha_n$ is a loop without an exit, then $r(\alpha_i)$ is a single vertex for all i.
We now extend Condition (L) to ultragraphs.

Condition (L): Every loop in $\mathcal{G}$ has an exit; that is, for any loop $\alpha := \alpha_1 \ldots \alpha_n$ there is either an edge $e \in \mathcal{G}^1$ such that $s(e) \in r(\alpha_i)$ and $e \neq \alpha_{i+1}$ for some i, or there is a sink w with $w \in r(\alpha_i)$ for some i.

A version of the Cuntz-Krieger Uniqueness Theorem for ultragraph algebras first appeared in [43, Theorem 6.1].

Theorem 4.9 (Cuntz-Krieger Uniqueness Theorem)**.** *Let $\mathcal{G}$ be an ultragraph satisfying Condition (L). If $\rho : C^*(\mathcal{G}) \to B$ is a $*$-homomorphism between C^*-algebras, and if $\rho(p_v) \neq 0$ for all $v \in G^0$, then ρ is injective.*

Note that if $\rho(p_v) \neq 0$ for all $v \in G^0$, then $\rho(p_A) \neq 0$ for all nonempty $A \in \mathcal{G}^0$, since p_A dominates p_v for all $v \in A$.

Furthermore, by the universal property for $C^*(\mathcal{G})$ there exists a gauge action $\gamma_z : \mathbb{T} \to \operatorname{Aut} C^*(\mathcal{G})$ with $\gamma_z(p_A) = p_A$ and $\gamma_z(s_e) = z s_e$ for all $A \in \mathcal{G}^0$ and $e \in G^1$. It is shown in [43, Theorem 6.2] that there is a Gauge-Invariant Uniqueness Theorem for ultragraph algebras.

Theorem 4.10 (Gauge-Invariant Uniqueness Theorem)**.** *Let $\mathcal{G}$ be an ultragraph, let $\{s_e, p_A\}$ the canonical generators in $C^*(\mathcal{G})$, and let γ the gauge action on $C^*(\mathcal{G})$. Also let B be a C^*-algebra, and $\rho : C^*(\mathcal{G}) \to B$ be a $*$-homomorphism for which $\rho(p_v) \neq 0$ for all $v \in G^0$. If there exists a strongly* *continuous action β of $\mathbb{T}$ on B such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$, then ρ is injective.*

Conditions for simplicity of an ultragraph algebra have been obtained in [44]. In order to state these conditions, one needs a notion of saturated hereditary collections.

Definition 4.11. A subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *hereditary* if

(1) whenever e is an edge with $\{s(e)\} \in \mathcal{H}$, then $r(e) \in \mathcal{H}$
(2) $A, B \in \mathcal{H}$, implies $A \cup B \in \mathcal{H}$
(3) $A \in \mathcal{H}$, $B \in \mathcal{G}^0$, and $B \subseteq A$, imply that $B \in \mathcal{H}$.

Definition 4.12. A hereditary subcollection $\mathcal{H} \subset \mathcal{G}^0$ is *saturated* if for any $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$ we have that

$$\{r(e) : e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq \mathcal{H} \quad \text{implies} \quad \{v\} \in \mathcal{H}.$$

Then [44, Theorem 3.10] states that an ultragraph algebra $\mathcal{G}$ is simple if and only if $\mathcal{G}$ satisfies Condition (L) and $\mathcal{G}^0$ contains no saturated hereditary subcollections other than $\emptyset$ and $\mathcal{G}^0$.

In addition, the dichotomy holds for simple ultragraph algebras; it is shown in [44, Proposition 4.5] that every simple ultragraph algebra is either AF or purely infinite.

Remark 4.13. In the forthcoming article [24] the collection $\mathcal{G}^0$ is defined to be the smallest subcollection of the power set of G^0 that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite intersections, finite unions, and relative complements (i.e. $A, B \in \mathcal{G}^0$ implies $A \setminus B \in \mathcal{G}^0$). Using this definition, one obtains the same C^*-algebra $C^*(\mathcal{G})$, however, this alternate definition is sometimes more convenient and allows one to avoid certain technicalities.

4.3. **Cuntz-Pimsner Algebras.** The Cuntz-Pimsner algebras are a vast generalization of graph algebras in which a C^*-algebra is associated to a C^*-correspondence (sometimes also called a Hilbert bimodule). In addition to graph algebras, Cuntz-Pimsner algebras generalize crossed products by $\mathbb{Z}$, ultragraph algebras, and many other well-known C^*-algebras.

Definition 4.14. If A is a C^*-algebra, then a *right Hilbert A-module* is a Banach space X together with a right action of A on X and an A-valued inner product $\langle \cdot, \cdot \rangle_A$ satisfying

(i) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$
(ii) $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$
(iii) $\langle \xi, \xi \rangle_A \geq 0$ and $\|\xi\| = \langle \xi, \xi \rangle_A^{1/2}$

for all $\xi, \eta \in X$ and $a \in A$. For a Hilbert A-module X we let $\mathcal{L}(X)$ denote the C^*-algebra of adjointable operators on X, and we let $\mathcal{K}(X)$ denote the closed two-sided ideal of compact operators given by

$$\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in X\}$$

where $\Theta^X_{\xi,\eta}$ is defined by $\Theta_{\xi,\eta}(\zeta) := \xi\langle\eta,\zeta\rangle_A$.

Definition 4.15. If A is a C^*-algebra, then a *C^*-correspondence* is a right Hilbert A-module X together with a $*$-homomorphism $\phi : A \to \mathcal{L}(X)$. We consider ϕ as giving a left action of A on X by setting $a \cdot x := \phi(a)x$.

Definition 4.16. If X is a C^*-correspondence over A, then a *Toeplitz representation* of X into a C^*-algebra B is a pair (ψ, π) consisting of a linear map $\psi : X \to B$ and a $*$-homomorphism $\pi : A \to B$ satisfying

(i) $\psi(\xi)^*\psi(\eta) = \pi(\langle\xi,\eta\rangle_A)$
(ii) $\psi(\phi(a)\xi) = \pi(a)\psi(\xi)$
(iii) $\psi(\xi a) = \psi(\xi)\pi(a)$

for all $\xi, \eta \in X$ and $a \in A$.
If (ψ, π) is a Toeplitz representation of X into a C^*-algebra B, we let $C^*(\psi, \pi)$ denote the C^*-algebra generated by $\psi(X) \cup \pi(A)$.

A Toeplitz representation (ψ, π) is said to be *injective* if π is injective. Note that in this case ψ will be isometric since

$$\|\psi(\xi)\|^2 = \|\psi(\xi)^*\psi(\xi)\| = \|\pi(\langle\xi,\xi\rangle_A)\| = \|\langle\xi,\xi\rangle_A\| = \|\xi\|^2.$$

Definition 4.17. For a Toeplitz representation (ψ, π) of a C^*-correspondence X on B there exists a $*$-homomorphism $\pi^{(1)} : \mathcal{K}(X) \to B$ with the property that

$$\pi^{(1)}(\Theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*.$$

Definition 4.18. For an ideal I in a C^*-algebra A we define

$$I^\perp := \{a \in A : ab = 0 \text{ for all } b \in I\}$$

and we refer to $I^\perp$ as the *annihilator of I in A*. If X is a C^*-correspondence over A, we define an ideal $J(X)$ of A by

$$J(X) := \phi^{-1}(\mathcal{K}(X)).$$

We also define an ideal J_X of A by

$$J_X := J(X) \cap (\ker\phi)^\perp.$$

Definition 4.19. If X is a C^*-correspondence over A, we say that a Toeplitz representation (ψ, π) is *coisometric on J_X* if

$$\pi^{(1)}(\phi(a)) = \pi(a) \qquad \text{for all } a \in J_X.$$

We say that a Toeplitz representation (ψ_X, π_A) which is coisometric on J_X is *universal* if whenever (ψ, π) is a Toeplitz representation of X into a C^*-algebra B which is coisometric J_X, then there exists a $*$-homomorphism $\rho_{(\psi,\pi)} : C^*(\psi_X, \pi_A) \to B$ with the property that $\psi = \rho_{(\psi,\pi)} \circ \psi_X$ and $\pi =$

$\rho_{(\psi,\pi)} \circ \pi_A$. That is, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "\psi_X"'] \arrow[drr, bend left, "\psi"] & & \\
& \mathcal{O}_X \arrow[r, dotted, "\rho_{(\psi,\pi)}"] & B \\
A \arrow[ur, "\pi_A"] \arrow[urr, bend right, "\pi"'] & &
\end{tikzcd}$$

Definition 4.20. If X is a C^*-correspondence over A, then the *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the C^*-algebra $C^*(\psi_X, \pi_A)$ where (ψ_X, π_A) is a universal Toeplitz representation of X which is coisometric on J_X.

Now that we have a definition of the Cuntz-Pimsner algebras $\mathcal{O}_X$, we shall describe how to view graph algebras as Cuntz-Pimsner algebras. In particular, if E is a directed graph we shall describe how to construct a C^*-correspondence $X(E)$ from E whose Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ is isomorphic to the graph algebra $C^*(E)$.

Example 4.21 (The Graph C^*-correspondence)*.* If $E = (E^0, E^1, r, s)$ is a graph, we define $A := C_0(E^0)$ and

$$X(E) := \{x : E^1 \to \mathbb{C} : \text{ the function } v \mapsto \sum_{\{f \in E^1 : r(f) = v\}} |x(f)|^2 \text{ is in } C_0(E^0)\ \}.$$

Then $X(E)$ is a C^*-correspondence over A with the operations

$$\begin{aligned}
(x \cdot a)(f) &:= x(f)a(r(f)) \text{ for } f \in E^1 \\
\langle x, y\rangle_A(v) &:= \sum_{\{f \in E^1 : r(f) = v\}} \overline{x(f)}y(f) \text{ for } f \in E^1 \\
(a \cdot x)(f) &:= a(s(f))x(f) \text{ for } f \in E^1
\end{aligned}$$

and we call $X(E)$ the *graph C^*-correspondence* associated to E. Note that we could write $X(E) = \bigoplus^0_{v \in E^0} \ell^2(r^{-1}(v))$ where this denotes the C_0 direct sum (sometimes called the restricted sum) of the $\ell^2(r^{-1}(v))$'s. Also note that $X(E)$ and A are spanned by the point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$, respectively.

Theorem 4.22 ([14, Proposition 12])**.** *If E is a graph and $X = X(E)$, then $\mathcal{O}_X \cong C^*(E)$. Furthermore, if (ψ_X, π_A) is a universal Toeplitz representation of X that is coisometric on J_X, then $\{\psi_X(\delta_e), \pi_A(\delta_v)\}$ is a universal Cuntz-Krieger E-family in $\mathcal{O}_X$.*

We now examine how properties of the graph relate to properties of the graph correspondence. We say that a C^*-correspondence is *full* if

$$\overline{\operatorname{span}}\{\langle x, y\rangle : x, y \in X\} = A,$$

and we say a C^*-correspondence is *essential* if

$$\overline{\operatorname{span}}\{\phi(a)x : x \in X \text{ and } a \in A\} = X.$$

It was shown in [16, Proposition 4.4] that

$$J(X(E)) = \overline{\operatorname{span}}\{\delta_v : |s^{-1}(v)| < \infty\}$$

and if v emits finitely many edges, then

$$\phi(\delta_v) = \sum_{\{f \in E^1 : s(f) = v\}} \Theta_{\delta_f, \delta_f} \quad \text{and} \quad \pi_A(\phi(\delta_v)) = \sum_{\{f \in E^1 : s(f) = v\}} \psi_X(\delta_f)\psi_X(\delta_f)^*.$$

Furthermore, one can see that $\delta_v \in \ker \phi$ if and only if v is a sink in E. Also $\delta_v \in \overline{\operatorname{span}}\{\langle x, y\rangle_A\}$ if and only if v is a source, and since $\delta_{s(f)} \cdot \delta_f = \delta_f$ we see that $\overline{\operatorname{span}}\, A \cdot X = X$ and $X(E)$ is essential. These observations show that we have the following correspondences between the properties of the graph E and the properties of the graph C^*-correspondence $X(E)$.

Property of $X(E)$	**Property of E**
$\phi(\delta_v) \in \mathcal{K}(X(E))$	v emits a finite number of edges
$\phi(A) \subseteq \mathcal{K}(X(E))$	E is row-finite
ϕ is injective	E has no sinks
$X(E)$ is full	E has no sources
$X(E)$ is essential	always

Remembering these properties will help us as we consider results for Cuntz-Pimsner algebras. For example, if X is a C^*-correspondence with $\phi(A) \subseteq \mathcal{K}(X)$, then the theory for $\mathcal{O}_X$ is similar to the theory for row-finite graph algebras. Likewise, if $\phi(A) \subseteq \mathcal{K}(X)$ and ϕ is injective, then the theory for $\mathcal{O}_X$ is similar to the theory for row-finite graph algebras with no sinks.

Remark 4.23. If E is a graph with no sinks, then $\phi(\delta_v) = 0$ if and only if v is a sink, and $\delta_v \in (\ker \phi)^\perp$ if and only if v is not a sink. Thus

$$J_{X(E)} = \overline{\operatorname{span}}\{\delta_v : 0 < |s^{-1}(v)| < \infty\}.$$

Remark 4.24. Suppose that $\mathcal{O}_X$ is a Cuntz-Pimsner algebra associated to a C^*-correspondence X, and that (ψ, π) is a universal Toeplitz representation of X which is coisometric on J_X. Then for any $z \in \mathbb{T}$ we have that $(z\psi, \pi)$ is also a universal Toeplitz representation which is coisometric on K. Hence by the universal property, there exists a homomorphism $\gamma_z : \mathcal{O}_X \to \mathcal{O}_X$ such that $\gamma_z(\pi(a)) = \pi(a)$ for all $a \in A$ and $\gamma_z(\psi(\xi)) = z\psi(\xi)$ for all $\xi \in X$. Since $\gamma_{z^{-1}}$ is an inverse for this homomorphism, we see that γ_z is an automorphism. Thus we have an action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_X$ with the property that $\gamma_z(\pi(a)) = \pi(a)$ and $\gamma_z(\psi(\xi)) = z\psi(\xi)$.

There exists a Gauge-Invariant Uniqueness Theorem for Cuntz-Pimsner algebras [22, Theorem 6.4].

Theorem 4.25 (Gauge-Invariant Uniqueness Theorem)**.** *Let X be a C^*-correspondence and let $\rho : \mathcal{O}_X \to B$ be a $*$-homomorphism between C^*-algebras with the property that $\rho|_{\pi_A(A)}$ is injective. If there exists a gauge action β of $\mathbb{T}$ on B such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$, then ρ is injective.*

In addition, the gauge-invariant ideals for a Cuntz-Pimsner algebra can be classified, in analogy with Theorem 1.6 and Theorem 2.24. As with graph algebras, this description takes the nicest form when $\phi(A) \subseteq \mathcal{K}(X)$ and ϕ is injective.

Definition 4.26. Let X be a C^*-correspondence over A. We say that an ideal $I \triangleleft A$ is X*-invariant* if $\phi(I)X \subseteq XI$. We say that an X-invariant ideal $I \triangleleft A$ is X*-saturated* if

$$a \in J_X \text{ and } \phi(a)X \subseteq XI \implies a \in I.$$

The next theorem follows from [28, Theorem 6.4] and [15, Corollary 3.3].

Theorem 4.27. *Let X be a C^*-correspondence with the property that $\phi(A) \subseteq X$ and ϕ is injective. Also let (ψ_X, π_A) be a universal Toeplitz representation of X that is coisometric on J_X. Then there is a lattice isomorphism from the X-saturated X-invariant ideals of A onto the gauge-invariant ideals of $\mathcal{O}_X$ given by*

$$I \mapsto \mathcal{I}(I) := \textit{the ideal in } \mathcal{O}_X \textit{ generated by } \pi_A(I).$$

Furthermore, $\mathcal{O}_X/\mathcal{I}(I) \cong \mathcal{O}_{X/XI}$, and the ideal $\mathcal{I}(I)$ is Morita equivalent to $\mathcal{O}_{XI}$.

For general C^*-correspondences the gauge-invariant ideals of $\mathcal{O}_X$ correspond to admissible pairs of ideals (I, J) coming from A. (See [23, Theorem 8.6] for more details.)

Although simplicity of $\mathcal{O}_X$ has been characterized for C^*-correspondences satisfying certain hypotheses, there is no general characterization of simplicity for $\mathcal{O}_X$. In addition, it is unknown whether there is an analogue of Condition (L) for C^*-correspondences, and currently there does not exist a Cuntz-Krieger Uniqueness Theorem for Cuntz-Pimsner algebras. It is also known that the dichotomy does not hold for Cuntz-Pimnser algebras: there are simple Cuntz-Pimsner algebras that are neither AF nor purely infinite.

In addition, a six-term exact sequence for the K-groups of $\mathcal{O}_X$ has been established in [18, Theorem 8.6] generalizing that of [33, Theorem 4.9]. This sequence allows one to calculate the K-theory of $\mathcal{O}_X$ in certain situations. It is a fact that all possible K-groups can be realized as the K-theory of Cuntz-Pimsner algebras.

4.4. **Topological Quiver Algebras.** Because the Cuntz-Pimsner algebras encompass such a wide class of C^*-algebras and exhibit a variety of behavior, it is sometimes difficult to study them in full generality. Therefore, authors will sometimes seek a "nice" subclass of Cuntz-Pimsner algebras whose behavior is similar to familiar C^*-algebras. One such subclass is the topological quiver algebras, which we will define by generalizing the construction of the graph C^*-correspondence described in Example 4.21. (We refer the reader to [29] for a more detailed exposition of topological quivers and their C^*-algebras.)

Definition 4.28. A *topological quiver* is a quintuple $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ consisting of a second countable locally compact Hausdorff space E^0 (whose elements are called vertices), a second countable locally compact Hausdorff space E^1 (whose elements are called edges), a continuous open map $r : E^1 \to E^0$, a continuous map $s : E^1 \to E^0$, and a family of Radon measures $\lambda = \{\lambda_v\}_{v \in E^0}$ on E^1 satisfying the following two conditions:

(1) $\operatorname{supp} \lambda_v = r^{-1}(v)$ for all $v \in E^0$
(2) $v \mapsto \int_{E^1} \xi(\alpha) d\lambda_v(\alpha)$ is an element of $C_c(E^0)$ for all $\xi \in C_c(E^1)$.

The term "quiver" was chosen because of the relation of the notion to ring theory where finite directed graphs are called quivers. In addition, we see that directed graphs are topological quivers in which the vertex and edge spaces have the discrete topology and the measure λ_v is counting measure for all vertices v.

We mention that if one is given E^0, E^1, r, and s as described in Definition 4.28, then there always exists a family of Radon measures $\lambda = \{\lambda_v\}_{v \in E^0}$ satisfying Conditions (1) and (2) (the existence relies on the fact that E^1 is second countable). However, in general this choice of λ is not unique.

When the map r is a local homeomorphism and λ_v is chosen as counting measure, we call the quiver a *topological graph*. Topological graphs have been studied extensively in [18, 19, 20, 21].

A topological quiver $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ gives rise to a C^*-correspondence in the following manner: We let $A := C_0(E^0)$ and define an A-valued inner product on $C_c(E^1)$ by

$$\langle \xi, \eta \rangle_A(v) := \int_{r^{-1}(v)} \overline{\xi(\alpha)} \eta(\alpha) \, d\lambda_v(\alpha) \qquad \text{for } v \in E^0 \text{ and } \xi, \eta \in C_c(E^1).$$

We shall let X denote the closure of $C_c(E^1)$ in the norm arising from this inner product. We define a right action of A on X by setting

$$\xi \cdot f(\alpha) := \xi(\alpha) f(r(\alpha)) \qquad \text{for } \alpha \in E^1,\ \xi \in C_c(E^1), \text{ and } f \in C_0(E^0)$$

and extending to all of X. We also define a left action $\phi : A \to \mathcal{L}(X)$ by setting

$$\phi(f)\xi(\alpha) := f(s(\alpha))\xi(\alpha) \qquad \text{for } \alpha \in E^1,\ \xi \in C_c(E^1), \text{ and } f \in C_0(E^0)$$

and extending to all of X. With this inner product and these actions X is a C^*-correspondence over A, and we refer to X as the *C^*-correspondence associated to $\mathcal{Q}$.*

Definition 4.29. If $\mathcal{Q}$ is a topological quiver, then we define $C^*(\mathcal{Q}) := \mathcal{O}_X$, where X is the C^*-correspondence associated to $\mathcal{Q}$. We let $(\psi_{\mathcal{Q}}, \pi_{\mathcal{Q}})$ denote the universal Toeplitz representation of X into $C^*(\mathcal{Q})$ that is coisometric on J_X.

Since $A := C_0(E^0)$ is a commutative C^*-algebra, it follows that the ideals of A correspond to open subsets of E^0. In the following definition we identify some of these subsets for important ideals associated with X.

Definition 4.30. If $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ is a topological quiver, we define the following:

(1) $E^0_{\text{sinks}} = E^0 \backslash \overline{s(E^1)}$
(2) $E^0_{\text{fin}} = \{v \in E^0$: there exists a precompact neighborhood V of v such that $s^{-1}(\overline{V})$ is compact and $r|_{s^{-1}(V)}$ is a local homeomorphism$\}$
(3) $E^0_{\text{reg}} := E^0_{\text{fin}} \backslash \overline{E^0_{\text{sinks}}}$

Remark 4.31. The notation and terminology of Definition 4.30 is meant to generalize the various types of vertices found in directed graphs. It can be shown that $\phi^{-1}(0) = C_0(E^0_{\text{sinks}})$, $\phi^{-1}(\mathcal{K}(X)) = C_0(E^0_{\text{fin}})$, $J_X = C_0(E^0_{\text{reg}})$. And when $\mathcal{Q}$ is a discrete graph, the sets E^0_{sinks}, E^0_{fin}, and E^0_{reg} correspond to the sinks, finite-emitters, and regular vertices (i.e., vertices that are neither sinks nor infinite emitters).

Because they are Cuntz-Pimsner algebras, quiver algebras have a natural gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(\mathcal{Q})$ with $\gamma_z(\pi_{\mathcal{Q}}(a)) = \pi_{\mathcal{Q}}(a)$ and $\gamma_z(\psi_{\mathcal{Q}}(x)) = z\psi_{\mathcal{Q}}(x)$ for $a \in A$ and $x \in X$. There is also a Gauge-Invariant Uniqueness Theorem for quiver algebras.

Theorem 4.32 (Guage-Invariant Uniqueness Theorem)**.** *Let $\mathcal{Q}$ be a topological quiver and let X be the C^*-correspondence over A associated to $\mathcal{Q}$. Let $\rho : C^*(\mathcal{Q}) \to B$ be a $*$-homomorphism between C^*-algebras with the property that $\rho|_{\pi_{\mathcal{Q}}(A)}$ is injective. If there exists a gauge action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$, then ρ is injective.*

In addition, the gauge-invariant ideals of $C^*(\mathcal{Q})$ can be described. In analogy with graph algebras, this takes the nicest form when $E^0_{\text{reg}} = E^0$.

Definition 4.33. Let $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ be a topological quiver. We say that a subset $U \subseteq E^0$ is *hereditary* if whenever $\alpha \in E^1$ and $s(\alpha) \in U$, then $r(\alpha) \in U$. We say that a hereditary subset U is *saturated* if whenever $v \in E^0_{\text{reg}}$ and $r(s^{-1}(v)) \subseteq U$, then $v \in U$.

Theorem 4.34. *Let $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ be a topological quiver with the property that $E^0_{\text{reg}} = E^0$. Then there is a bijective correspondence from the set of saturated hereditary open subsets of E^0 onto the gauge-invariant ideals of $C^*(\mathcal{Q})$ given by*

$$U \mapsto \mathcal{I}_U := \text{ the ideal in } C^*(\mathcal{Q}) \text{ generated by } \pi_{\mathcal{Q}}(C_0(U)).$$

Furthermore, for any saturated hereditary open subset U we have that $\mathcal{I}_U$ is Morita equivalent to $C^*(\mathcal{Q}_U)$, where $\mathcal{Q}_U$ is the subquiver of $\mathcal{Q}$ whose vertices *are U and whose edges are $s^{-1}(U)$, and $C^*(\mathcal{Q})/\mathcal{I}_U \cong C^*(\mathcal{Q}\setminus U)$, where $\mathcal{Q}\setminus U$ is the subquiver of $\mathcal{Q}$ whose vertices are $E^0 \setminus U$ and edges are $E^1 \setminus r^{-1}(U)$.*

For general topological quivers, the gauge-invariant ideals of $C^*(\mathcal{Q})$ correspond to pairs (U, V) of admissible subsets. (See [29, §8] for more details.)

In addition there is a version of Condition (L), and a Cuntz-Krieger Uniqueness Theorem for quiver algebras. Note that Condition (L) makes use of the topology on E^0.

Condition (L): The set of base points of loops in $\mathcal{Q}$ with no exits has empty interior.

Theorem 4.35 (Cuntz-Krieger Uniqueness Theorem)**.** *Let $\mathcal{Q}$ be a topological quiver that satisfies Condition (L), and let X be the C^*-correspondence over A associated to $\mathcal{Q}$. If $\rho : C^*(\mathcal{Q}) \to B$ is a $*$-homomorphism from $C^*(\mathcal{Q})$ into a C^*-algebra B with the property that the restriction $\rho|_{\pi_{\mathcal{Q}}(A)}$ is injective, then ρ is injective.*

Furthermore, simplicity of quiver algebras has been characterized: The quiver algebra $C^*(\mathcal{Q})$ is simple if and only if Q satisfies Condition (L) and there are no saturated hereditary open subsets of E^0 other than $\emptyset$ and E^0 [29, Theorem 10.2].

We mention also that the dichotomy does not hold for quiver algebras: There are simple quiver algebras that are neither AF nor purely infinite.

Also, a there is version of Condition (K) for quiver algebras.

Definition 4.36. If $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ is a topological quiver and $v, w \in E^0$, then we write $w \geq v$ to mean that there is a path $\alpha \in E^n$ with $s(\alpha) = w$ and $r(\alpha) = v$. We also define $v^{\geq} := \{w \in E^0 : w \geq v\}$.

Condition (K): The set

$$\{v \in E^0 : v \text{ is the base point of exactly one simple loop} \\ \text{and } v \text{ is isolated in } v^{\geq}\ \}$$

is empty.

Theorem 4.37. *(*[29, Theorem 9.10]*) Let $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ be a topological quiver that satisfies Condition (K). Then every ideal in $C^*(\mathcal{Q})$ is gauge invariant.*

Remark 4.38. It has been shown by Katsura that every AF algebra is isomorphic to the C^*-algebra of a topological graph, and that every Kirchberg-Phillips algebra is isomorphic to the C^*-algebra of a topological graph. In addition, in a forthcoming paper of Katsura, Muhly, Sims, and Tomforde it will be shown that every ultragraph algebra is the C^*-algebra of a topological graph. Hence the class of quiver algebras contains all ultragraph algebras and also all Exel-Laca algebras. Furthermore, the only known conditions that a topological graph algebra must satisfy are: (1) it must be nuclear, and (2) it must satisfy the UCT. At the time of this writing it is an open question whether any nuclear C^*-algebra satisfying the UCT is isomorphic to a topological graph algebra.

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